Enlargements of Filtrations with Finance in view

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These notes are not complete, and have to be considered as a preliminary version of a full text. The first version was written to give a support to the lectures given by Monique Jeanblanc and Giorgia Callegaro at University El Manar, Tunis, in March 2010 and later by Monique Jeanblanc in various schools: Jena, in June 2010, Beijing, for the Sino-French Summer institute, June 2011 and as a main course in Master 2, Marne La Vallée, Winter 2011.

Paragraphs marked with ✓ have to be completed
Some solutions of exercices are written by Giorgia Callegaro.

A large part of these notes can be found in the book

Mathematical Methods in Financial Markets,

by Jeanblanc, M., Yor M., and Chesney, M. (Springer), indicated as [3M]. We thank Springer Verlag for allowing us to reproduce part of this volume.

Many examples related to Credit risk can be found in

Credit Risk Modeling

by Bielecki, T.R., Jeanblanc, M. and Rutkowski, M., CSFI lecture note series, Osaka University Press, 2009, indicated as [BJR].

I thank all the participants to Tunis lecture, Jena workshop and Beijing summer school for their remarks, questions and comments.

Contents

Chapter 1

Theory of Stochastic Processes

In this chapter, we recall some facts on theory of stochastic processes. Proofs can be found for example in Dellacherie [34], Dellacherie and Meyer [38], He, Wang and Yan [59] and Rogers and Williams [102].

1.1 Background

As usual, we start with a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$ is a given filtration satisfying the usual conditions, i.e., $\mathbb{F}$ is continuous on right ($\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$) and $\mathcal{F}_0$ contains all negligeable sets, and $\mathcal{F} = \mathcal{F}_\infty$. A process X is a family of random variables such that $(\omega, t) \to X_t(\omega)$ is $\mathcal{F} \times \mathcal{B}$ measurable, where $\mathcal{B}$ is the Borel field on $\mathbb{R}^+$ (one says also measurable process).

1.1.1 Path properties

Definition 1.1.1 *1) A process X is **continuous** if, for almost all ω, the map $t \to X_t(\omega)$ is continuous. A process X is continuous on the right with limits on the left (in short **càdlàg** following the French acronym[1]) if, for almost all ω, the map $t \to X_t(\omega)$ is càdlàg.*
*2) A process A is **increasing** if $A_0 = 0$, A is right-continuous, and $A_s \leq A_t$, a.s. for $s \leq t$. An increasing process $A = (A_t, t \geq 0)$ is integrable if $E(A_{\infty-}) < \infty$.*

Sometimes, one has to consider increasing processes defined for $t \in [0, \infty]$ (with a possible jump at $+\infty$). In that case, the process is integrable if $E(A_\infty) < \infty$.
For a (right-continuous) increasing process, we note $\int_a^b \varphi_s dA_s := \int_{]a,b]} \varphi_s dA_s$ as soon as the integral is well defined. The point here is that the integration is done on the interval $]a, b]$

Definition 1.1.2 *A process X is $\mathbb{F}$-adapted if for any $t \geq 0$, the random variable X_t is $\mathcal{F}_t$-measurable.*

The **natural filtration** $\mathbb{F}^X$ of a stochastic process X is the smallest filtration $\mathbb{F}$ which satisfies the usual hypotheses and such that X is $\mathbb{F}$-adapted. We shall write in short (with an abuse of notation) $\mathcal{F}_t^X = \sigma(X_s, s \leq t)$.

Remark 1.1.3 It is not true in general that if $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ are right-continuous, the filtration $\mathbb{K}$ defined as $\mathcal{K}_t := \mathcal{F}_t \vee \widetilde{\mathcal{F}}_t$ is right-continuous.

[1] In French, continuous on the right is continu à droite, and with limits on the left is admettant des limites à gauche. We shall also use càd for continuous on the right. The use of this acronym comes from P-A. Meyer.

Exercise 1.1.4 Starting from a non continuous on right filtration $\mathbb{F}^0$, define the smallest right-continuous filtration $\mathbb{F}$ which contains $\mathbb{F}^0$. $\triangleleft$

In all the book, we shall write $X \in \mathcal{F}_T$ (resp. $X \in b\mathcal{F}_T$) for X is an $\mathcal{F}_T$-measurable (resp. a bounded $\mathcal{F}_T$-measurable) random variable.

1.1.2 Stopping times

A random variable τ, valued in $[0, \infty]$ is an $\mathbb{F}$ stopping time if, for any $t \geq 0$, $\{\tau \leq t\} \in \mathcal{F}_t$.

A stopping time τ is **predictable** if there exists an increasing sequence (τ_n) of stopping times such that almost surely

(i) $\lim_n \tau_n = \tau$,

(ii) $\tau_n < \tau$ for every n on the set $\{\tau > 0\}$. If needed, we shall make precise the choice of the filtration, writing that the $\mathbb{F}$ stopping time τ is $\mathbb{F}$-predictable.

A stopping time τ is **totally inaccessible** if $\mathbb{P}(\tau = \vartheta < \infty) = 0$ for any predictable stopping time ϑ (or, equivalently, if for any increasing sequence of stopping times $(\tau_n, n \geq 0)$, $\mathbb{P}(\{\lim \tau_n = \tau\} \cap A) = 0$ where $A = \cap_n \{\tau_n < \tau\}$).

If all $\mathbb{F}$ martingales are continuous, then any $\mathbb{F}$ stopping time is predictable. THis is the case in particular if $\mathbb{F}$ is a Brownian filtration

Definition 1.1.5 *If τ is an $\mathbb{F}$-stopping time, the σ-algebra $\mathcal{F}_\tau$ of events prior to τ, and the σ-algebra $\mathcal{F}_{\tau-}$ of events strictly prior to τ are defined as:*

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty \,:\, A \cap \{\tau \leq t\} \in \mathcal{F}_t, \, \forall t\}$$

whereas $\mathcal{F}_{\tau-}$ is the smallest σ-algebra which contains $\mathcal{F}_0$ and all the sets of the form $A \cap \{t < \tau\}, t > 0$ for $A \in \mathcal{F}_t$.

For $A \in \mathcal{F}_\tau$, one sets τ_A the stopping time defined as $\tau_A = \tau \mathbb{1}_A + \infty \mathbb{1}_{A^c}$.

Exercise 1.1.6 Prove that τ_A is a stopping time. $\triangleleft$

Exercise 1.1.7 Show that for a stopping time τ, one has $\tau \in \mathcal{F}_{\tau-}$ and $\mathcal{F}_{\tau-} \subset \mathcal{F}_\tau$. Find an example where $\mathcal{F}_{\tau-} \neq \mathcal{F}_\tau$ $\triangleleft$

Exercise 1.1.8 Check that if $\mathbb{F} \subset \mathbb{G}$ and τ is an $\mathbb{F}$-stopping time, (resp. $\mathbb{F}$-predictable stopping time) it is a $\mathbb{G}$-stopping time, (resp. $\mathbb{G}$-predictable stopping time). Give an example where τ is a $\mathbb{G}$-stopping time but not an $\mathbb{F}$ stopping time. Give an example where τ is a $\mathbb{G}$-predictable stopping time, and an $\mathbb{F}$ stopping time, but not a predictable $\mathbb{F}$-stopping time. $\triangleleft$

1.1.3 Predictable and optional σ-algebra

If τ and ϑ are two stopping times, the **stochastic interval** $]\!]\vartheta, \tau]\!]$ is the set $\{(\omega, t) : \vartheta(\omega) < t \leq \tau(\omega)\}$. In the same way, we shall use the notation $[\![\vartheta, \tau]\!]$, as well as for other stochastic intervals.

If necessary, we shall note $\mathcal{P}(\mathbb{F})$ this predictable σ-algebra, to emphasize the rôle of $\mathbb{F}$.

Proposition 1.1.9 *Let $\mathbb{F}$ be a given filtration.*

- *The* **optional** *σ-algebra $\mathcal{O}$ is the σ-algebra on $\mathbb{R}^+ \times \Omega$ generated by càdlàg $\mathbb{F}$-adapted processes (considered as mappings on $\mathbb{R}^+ \times \Omega$). The optional σ-algebra $\mathcal{O}$ is equal to the σ-algebra generated on $\mathcal{F} \times \mathcal{B}$ by the stochastic intervals $[\![\tau, \infty[\![$ where τ is an $\mathbb{F}$-stopping time.*

- *The* **predictable** *σ-algebra $\mathcal{P}$ is the σ-algebra on $\mathbb{R}^+ \times \Omega$ generated by the $\mathbb{F}$-adapted càg (or continuous) processes. The predictable σ-algebra $\mathcal{P}$ is equal to the σ-algebra generated on $\mathcal{F} \times \mathcal{B}$ by the stochastic intervals $]\!]\vartheta, \tau]\!]$ where ϑ and τ are two $\mathbb{F}$-stopping times such that $\vartheta \le \tau$.*

A process X is said to be $\mathbb{F}$-**predictable** (resp. $\mathbb{F}$-**optional**) if the map $(\omega, t) \to X_t(\omega)$ is $\mathcal{P}$-measurable (resp. $\mathcal{O}$-measurable).

Example 1.1.10 An adapted càg process is predictable.

The inclusion $\mathcal{P} \subset \mathcal{O}$ holds. These two σ-algebras $\mathcal{P}$ and $\mathcal{O}$ are equal if all $\mathbb{F}$-martingales are continuous. Note that $\mathcal{O} = \mathcal{P}$ if and only if any stopping time is predictable. In general

$$\mathcal{O} = \mathcal{P} \vee \sigma(\Delta M, M \text{ describing the set of } \mathbb{F} \text{ martingales}) .$$

If X is a predictable (resp. optional) process and τ a stopping time, then the stopped process $X^\tau = (X^\tau_t = X_{t\wedge\tau}, t \ge 0)$ is also predictable (resp. optional). If X is a càdlàg adapted process, then $(X_{t^-}, t \ge 0)$ is a predictable process.

If τ is a stopping time, the (càg) process $1\!\!1_{\tau<t}$ is predictable. A stopping time τ is **predictable** if and only if the process $(1\!\!1_{\{t<\tau\}} = 1 - 1\!\!1_{\{\tau \le t\}}, t \ge 0)$ is predictable, that is if and only if the stochastic interval $[\![0, \tau[\![= \{(\omega, t) : 0 \le t < \tau(\omega)\}$ is predictable. See Dellacherie [34], Dellacherie and Meyer [36] and Elliott [43] for related results.

Definition 1.1.11 *A real-valued process X is* ***progressively measurable*** *with respect to a given filtration $\mathbb{F} = (\mathcal{F}_t, t \ge 0)$, if, for every t, the map $(\omega, s) \to X_s(\omega)$ from $\Omega \times [0, t]$ into $\mathbb{R}$ is $\mathcal{F}_t \times \mathcal{B}([0, t])$-measurable.*

Any càd (or càg) $\mathbb{F}$-adapted process is progressively measurable. An $\mathbb{F}$-progressively measurable process is $\mathbb{F}$-adapted. If X is progressively measurable, then

$$\mathbb{E}\left(\int_0^\infty X_t dt\right) = \int_0^\infty \mathbb{E}(X_t)\, dt,$$

where the existence of one of these expressions implies the existence of the other.
If X is $\mathbb{F}$-progressively measurable and τ an $\mathbb{F}$-stopping time, then the r.v. X_τ is $\mathcal{F}_\tau$-measurable on the set $\{\tau < \infty\}$.

If τ is a random time (i.e. a non negative r.v.), the σ-algebra $\mathcal{F}_\tau$ and $\mathcal{F}_{\tau-}$ are defined as

$$\begin{aligned}
\mathcal{F}_\tau &= \sigma(Y_\tau,\, Y \text{is an}\, \mathbb{F} - \text{optional process}) \\
\mathcal{F}_{\tau-} &= \sigma(Y_\tau,\, Y \text{is an}\, \mathbb{F} - \text{predictable process})
\end{aligned}$$

1.1.4 Localization

Definition 1.1.12 *An adapted, right-continuous process M is an $\mathbb{F}$-**local martingale** if there exists a sequence of stopping times (τ_n) such that:*

- *The sequence τ_n is increasing and $\lim_n \tau_n = \infty$, a.s.*
- *For every n, the stopped process $M^{\tau_n} 1\!\!1_{\{\tau_n>0\}}$ is an $\mathbb{F}$-martingale (we recall that $M^\tau_t = M_{t\wedge\tau}$).*

A sequence of stopping times such that the two previous conditions hold is called a localizing or reducing sequence. We also use the following definitions: A local martingale M is locally square integrable if there exists a localizing sequence of stopping times (τ_n) such that $M^{\tau_n} 1\!\!1_{\{\tau_n>0\}}$ is a square integrable martingale. An increasing process A is locally integrable if there exists a localizing sequence of stopping times such that A^{τ_n} is integrable. By similar localization, we may define locally

bounded martingales, local super-martingales, and locally finite variation processes.
If M is a local martingale, it is always possible to choose the localizing sequence $(\tau_n, n \geq 1)$ such that each martingale $M^{\tau_n} \mathbb{1}_{\{\tau_n > 0\}}$ is uniformly integrable.
We denote by $\mathcal{M}_{loc}(\mathbb{P}, \mathbb{F})$ the space of $\mathbb{P}$ local martingales relative to $\mathbb{F}$.

Exercise 1.1.13 Prove that a positive local martingale is a super-martingale. $\triangleleft$

1.1.5 Doob-Meyer decomposition

An adapted process X is said to be of class[2] (D) if the collection $X_\tau \mathbb{1}_{\tau<\infty}$ where τ is a stopping time is uniformly integrable.

If Z is a supermartingale of class (D), there exists a unique increasing, integrable and predictable process A such that $Z_t = \mathbb{E}(A_\infty - A_t | \mathcal{F}_t)$. In particular, any supermartingale of class (D) can be written as $Z_t = M_t - A_t$ where M is a uniformly integrable martingale. The decomposition is unique.

Any supermartingale can be written as $Z_t = M_t - A_t$ where M is a local martingale and A a predictable increasing process. The decomposition is unique.

There are other decompositions of semi-martingales, as a sum of a martingale and an optionnel process that are useful. We shall comment that later on.

Multiplicative decomposition of positive supermartingales

Let Z be a positive supermartingale. There exists a local martingale N and a predictable increasing process D such that $Z = Ne^{-D}$.

1.2 Semi-martingales

1.2.1 Definition

An $\mathbb{F}$-adapted process X is an $\mathbb{F}$-semi-martingale if $X = M + A$ where M is an $\mathbb{F}$-local martingale and A an $\mathbb{F}$-adapted process with finite variation. If there exists a decomposition with a process A which is predictable, the decomposition $X = M + A$ where M is an $\mathbb{F}$-martingale and A an $\mathbb{F}$-predictable process with finite variation is unique and X is called a **special** semi-martingale. If X is continuous, the process A is continuous.

In general, if $\mathbb{G} = (\mathcal{G}_t, t \geq 0)$ is a filtration larger than $\mathbb{F} = (\mathcal{F}_t, t \geq 0)$, i.e., $\mathcal{F}_t \subset \mathcal{G}_t, \forall t \geq 0$ (we shall write $\mathbb{F} \subset \mathbb{G}$), it is not true that an $\mathbb{F}$-martingale remains a martingale in the filtration $\mathbb{G}$. It is not even true that $\mathbb{F}$-martingales remain $\mathbb{G}$-semi-martingales. One of the goal of this book is to give conditions so that this property holds.

Example 1.2.1 (a) Let $\mathcal{G}_t = \mathcal{F}_\infty$. Then, the only $\mathbb{F}$-martingales which are $\mathbb{G}$-martingales are constants.
(b) An interesting example is Azéma's martingale μ, defined as follows. Let B be a Brownian motion and $g_t = \sup\{s \leq t, B_s = 0\}$. The process

$$\mu_t = (\text{sgn} B_t) \sqrt{t - g_t},\ t \geq 0$$

is a martingale in its own filtration. This discontinuous $\mathbb{F}^\mu$-martingale is not an $\mathbb{F}^B$-martingale, it is not even an $\mathbb{F}^B$-semi-martingale.

[2] Class (D) is in honor of Doob.

Exercise 1.2.2 Let B be a Brownian motion. Prove that $W_t = \int_0^t \text{sgne} B_s dB_s$ defines an $\mathbb{F}^B$ and an $\mathbb{F}^W$ Brownian motion.
Prove that $\beta_t = B_t - \int_0^t \frac{B_s}{s} ds$ defines a Brownian motion (in its own filtration) which is not a Brownian motion in $\mathbb{F}^B$. ◁

1.2.2 Properties

Proposition 1.2.3 *Let $\mathbb{G}$ be a filtration larger than $\mathbb{F}$, i.e., $\mathbb{F} \subset \mathbb{G}$. If x is a u.i. (uniformly integrable) $\mathbb{F}$-martingale, then there exists a $\mathbb{G}$-martingale X, such that $\mathbb{E}(X_t|\mathcal{F}_t) = x_t$, $t \geq 0$.*

Proof: The process X defined by $X_t := \mathbb{E}(x_\infty|\mathcal{G}_t)$ is a $\mathbb{G}$-martingale, and

$$\mathbb{E}(X_t|\mathcal{F}_t) = \mathbb{E}\big(\mathbb{E}(x_\infty|\mathcal{G}_t)|\mathcal{F}_t\big) = \mathbb{E}(x_\infty|\mathcal{F}_t) = x_t \ .$$

□

The uniqueness of such a martingale X is not claimed in the above proposition and it is not true in general.

We recall an important (but difficult) result due to Stricker [108].

Proposition 1.2.4 *Let $\mathbb{F}$ and $\mathbb{G}$ be two filtrations such that $\mathbb{F} \subset \mathbb{G}$. If X is a $\mathbb{G}$-semimartingale which is $\mathbb{F}$-adapted, then it is also an $\mathbb{F}$-semimartingale.*

One has also the (obvious) following result (see Exercise 1.2.9)

Proposition 1.2.5 *Let $\mathbb{F}$ and $\mathbb{G}$ be two filtrations such that $\mathbb{F} \subset \mathbb{G}$. If X is a $\mathbb{G}$-martingale which is $\mathbb{F}$-adapted, then it is also an $\mathbb{F}$-martingale.*

Remark 1.2.6 This result does not extend to local martingales. See Stricker [108] and Föllmer and Protter [53].

Exercise 1.2.7 Let N be a Poisson process (i.e., a process with stationary and independent increments, such that the law of N_t is a Poisson law with parameter λt). Prove that the process M defined as $M_t = N_t - \lambda t$ is a martingale and that the process $M_t^2 - \lambda t = (N_t - \lambda t)^2 - \lambda t$ is also a martingale. Prove that for any $\theta \in [0, 1]$,

$$N_t = \theta(N_t - \lambda t) + (1 - \theta)N_t + \theta\lambda t = \mu_t + (1 - \theta)N_t + \theta\lambda t$$

is a decomposition of the semi-martingale N, where μ is a martingale. For which decomposition is the finite variation process $(1 - \theta)N_t + \theta\lambda t$ a predictable process ? ◁

Exercise 1.2.8 Let τ be a random time. Prove that τ is a $\mathbb{H}$-stopping time, where $\mathbb{H}$ is the natural filtration of $H_t = 1\!\!1_{\{\tau \leq t\}}$, and that τ is a $\mathbb{G}$ stopping time, where $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, for any filtration $\mathbb{F}$. ◁

Exercise 1.2.9 Prove that, if M is a $\mathbb{G}$-martingale, then $\widehat{M}$ defined as $\widehat{M}_t = \mathbb{E}(M_t|\mathcal{F}_t)$ is an $\mathbb{F}$-martingale. ◁

Exercise 1.2.10 Prove that, if $\mathbb{G} = \mathbb{F} \vee \widetilde{\mathbb{F}}$ where $\widetilde{\mathbb{F}}$ is independent of $\mathbb{F}$, then any $\mathbb{F}$ martingale remains a $\mathbb{G}$-martingale.
Prove that, if $\mathbb{F}$ is generated by a Brownian motion W, and if there exists a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that $\widetilde{\mathbb{F}}$ is independent of $\mathbb{F}$ under $\mathbb{Q}$, then any $(\mathbb{P}, \mathbb{F})$-martingale remains a $(\mathbb{P}, \mathbb{G})$-semi martingale. ◁

1.2.3 Stochastic Integration

If $X = M + A$ is a semi-martingale and Y a (bounded) predictable process, we denote $Y\star X$ the stochastic integral

$$(Y\star X)_t := \int_0^t Y_s dX_s = \int_0^t Y_s dM_s + \int_0^t Y_s dA_s$$

The process $Y\star X$ is a semi-martingale.

1.2.4 Integration by Parts

By definition, any semi-martingale X admits a decomposition as a local martingale M and a finite variation process. The martingale part admits a decomposition as $M = M^c + M^d$ where M^c is continuous and M^d a discontinuous martingale. The process M^c is denoted in the literature as X^c (even if this notation is missleading!). The optional Itô formula is (for f in C^2, with bounded derivatives)

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t f'(X_{s^-})dX_s + \tfrac{1}{2}\int_0^t f''(X_s)d\langle X^c\rangle_s \\ &\quad + \textstyle\sum_{0<s\leq t}[f(X_s) - f(X_{s^-}) - f'(X_{s^-})\Delta X_s]\,. \end{aligned}$$

where $\Delta X_t = X_t - X_{t-}$ [3].

If U and V are two finite variation processes, the Stieltjes integration by parts formula can be written as follows:

$$\begin{aligned} U_tV_t &= U_0V_0 + \int_{]0,t]} V_s dU_s + \int_{]0,t]} U_{s^-} dV_s \qquad (1.2.1) \\ &= U_0V_0 + \int_{]0,t]} V_{s^-} dU_s + \int_{]0,t]} U_{s^-} dV_s + \sum_{s\leq t} \Delta U_s\, \Delta V_s\,. \end{aligned}$$

As a partial check, one can verify that the jump process of the left-hand side, i.e., $U_tV_t - U_{t^-}V_{t^-}$, is equal to the jump process of the right-hand side, i.e., $V_{t^-}\Delta U_t + U_{t^-}\Delta V_t + \Delta U_t\,\Delta V_t$.

Let X be a continuous local martingale. The predictable quadratic variation process of X is the continuous increasing process $\langle X\rangle$ such that $X^2 - \langle X\rangle$ is a local martingale.
Let X and Y be two continuous local martingales. The predictable covariation process is the continuous finite variation process $\langle X, Y\rangle$ such that $XY - \langle X, Y\rangle$ is a local martingale. The covariation process of continuous martingales does not depend on the filtration.

Let X and Y be two local martingales. The covariation process $[X, Y]$ is the finite variation process such that
(i) $XY - [X, Y]$ is a local martingale
(ii) $\Delta[X, Y]_t = \Delta X_t \Delta Y_t$
The process $[X] = [X, X]$ is non-decreasing;if X is continuous, then $[X] = \langle X\rangle$.
The predictable covariation process is (if it exists) the predictable finite variation process $\langle X, Y\rangle$ such that $XY - \langle X, Y\rangle$ is a local martingale.
If X is a semi-martingale with respect to $\mathbb{F}$ and to $\mathbb{G}$, then $[X]$ is independent of the filtration.

The integration by parts for semi-martingales is

$$X_tY_t = X_0Y_0 + \int_0^t X_{s-}dY_s + \int_0^t Y_{s-}dX_s + [X, Y]_t\,. \qquad (1.2.2)$$

For finite variation processes

$$[U, V]_t = \sum_{s\leq t} \Delta U_s\, \Delta V_s$$

[3]one can prove that, for a semi-martingale X, the sum is well defined.

and, if Y is with finite variation, Yoeurp's formula states that

$$X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_{s-} dX_s \,. \tag{1.2.3}$$

If M is a bounded martingale and A a predictable increasing process,

$$\mathbb{E}(M_\infty A_\infty) = \mathbb{E}(\int_0^\infty M_{s-} dA_s)\,.$$

Exercise 1.2.11 Prove that if X and Y are continuous, $\langle X, Y\rangle = [X, Y]$.
Prove that if M is the compensated martingale of a Poisson process with intensity λ, $[M] = N$ and $\langle M\rangle_t = \lambda t$. ◁

1.3 Change of probability and Girsanov's Theorem

1.3.1 Brownian filtration

Let $\mathbb{F}$ be a Brownian filtration, L an $\mathbb{F}$-martingale, strictly positive such that $L_0 = 1$ and define $d\mathbb{Q}|_{\mathcal{F}_t} = L_t d\mathbb{P}|_{\mathcal{F}_t}$. Then,

$$\widetilde{B}_t := B_t - \int_0^t \frac{1}{L_s} d\langle B, L\rangle_s$$

is a $(\mathbb{Q}, \mathbb{F})$-Brownian motion. If M is an $\mathbb{F}$-martingale,

$$\widetilde{M}_t := M_t - \int_0^t \frac{1}{L_s} d\langle M, L\rangle_s$$

is a $(\mathbb{Q}, \mathbb{F})$-local martingale.

1.3.2 Doléans-Dade exponential

Let $\mathbb{F}$ be a Brownian filtration and ψ an adapted process satisfying $\int_0^t \psi_s^2 ds < \infty, \forall t$. The solution of $dL_t = L_t\psi_t dW_t$ is the local martingale

$$L_t = L_0 \exp\left(\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t \psi_s^2 ds\right)$$

If $\mathbb{E}(L_t) = 1$, the processus L is a martingale.

If L is a strict local martingale, the positive measure $\mathbb{Q}$ defined as $d\mathbb{Q} = L_t d\mathbb{P}$ is not a probability ($\mathbb{Q}(\Omega) \neq 1$)

For a continuous martingale M, the solution of $dL_t = L_t\psi_t dM_t$ is a positive local martingale

$$L_t = L_0 \exp\left(\int_0^t \psi_s dM_s - \frac{1}{2}\int_0^t \psi_s^2 d\langle M\rangle_s\right)$$

If M is a martingale, the solution f $dL_t = L_t\psi_t dM_t$ can take negative values and Q is a signed measure.

1.3.3 General case

More generally, let $\mathbb{F}$ be a filtration and L an $\mathbb{F}$-martingale, strictly positive such that $L_0 = 1$ and define $d\mathbb{Q}|_{\mathcal{F}_t} = L_t d\mathbb{P}|_{\mathcal{F}_t}$. Then, if M is an $\mathbb{F}$-martingale,

$$\widetilde{M}_t := M_t - \int_0^t \frac{1}{L_s} d[M, L]_s$$

is a $(\mathbb{Q}, \mathbb{F})$-martingale. If the predictable co-variation process $\langle M, L\rangle$ exists,

$$M_t - \int_0^t \frac{1}{L_{s-}} d\langle M, L\rangle_s$$

is a $(\mathbb{Q}, \mathbb{F})$-local martingale.

1.3.4 Itô-Kunita-Wentcell formula

We recall here the Itô-Kunita-Wentcell formula (see Kunita [87]). Let $F_t(x)$ be a family of stochastic processes, continuous in $(t, x) \in (\mathbb{R}_+ \times \mathbb{R}^d)$ a.s., and satisfying the following conditions:
(i) for each $t > 0$, $x \to F_t(x)$ is C^2 from $\mathbb{R}^d$ to $\mathbb{R}$,
(ii) for each x, $(F_t(x), t \geq 0)$ is a continuous semimartingale

$$dF_t(x) = \sum_{j=1}^n f_t^j(x)\, dM_t^j ,$$

where M^j are continuous semimartingales, and $f^j(x)$ are stochastic processes continuous in (t, x), such that for every $s > 0$, the map $x \to f_s^j(x)$ is C^1, and for every x, $f^j(x)$ is an adapted process. Let $X = (X^1, \cdots, X^d)$ be a continuous semimartingale. Then

$$\begin{aligned} F_t(X_t) &= F_0(X_0) + \sum_{j=1}^n \int_0^t f_s^j(X_s)\, dM_s^j + \sum_{i=1}^d \int_0^t \frac{\partial F_s}{\partial x_i}(X_s)\, dX_s^i \\ &+ \sum_{i=1}^d \sum_{j=1}^n \int_0^t \frac{\partial f_s}{\partial x_i}(X_s)\, d\langle M^j, X^i\rangle_s + \frac{1}{2} \sum_{i,k=1}^d \int_0^t \frac{\partial^2 F_s}{\partial x_i \partial x_k}\, d\langle X^k, X^i\rangle_s . \end{aligned}$$

See Bank and Baum [13] for an extension to processes with jumps.

✓Existence of bracket, Tanaka, local time

1.4 Projections and Dual Projections

In this section, after recalling some basic facts about optional and predictable projections, we introduce the concept of a dual predictable (resp. optional) projection, which leads to the fundamental notion of predictable compensators. We recommend the survey paper of Nikeghbali [97].

1.4.1 Definition of Projections

Let X be a bounded (or positive) process, and $\mathbb{F}$ a given filtration (we do not assume that X is $\mathbb{F}$-adapted). The **optional projection** of X is the unique optional process ${}^{(o)}X$ which satisfies: for any $\mathbb{F}$-stopping time τ

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}}) = \mathbb{E}({}^{(o)}X_\tau \mathbb{1}_{\{\tau<\infty\}}) . \tag{1.4.1}$$

In case where many filtrations are involved, we shall use the notation ${}^{(o,\mathbb{F})}X$ for the $\mathbb{F}$ optional projection. For any $\mathbb{F}$-stopping time τ, let $\Gamma \in \mathcal{F}_\tau$ and apply the equality (1.4.1) to the stopping time $\tau_\Gamma = \tau \mathbb{1}_\Gamma + \infty \mathbb{1}_{\Gamma^c}$. We get the re-inforced identity:

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}} | \mathcal{F}_\tau) = {}^{(o)}X_\tau \mathbb{1}_{\{\tau<\infty\}} .$$

In particular, if A is an increasing process, then, for $s \leq t$:

$$\mathbb{E}({}^{(o)}A_t - {}^{(o)}A_s | \mathcal{F}_s) = \mathbb{E}(A_t - A_s | \mathcal{F}_s) \geq 0 . \tag{1.4.2}$$

Note that, for any t, $\mathbb{E}(X_t|\mathcal{F}_t) = {}^{(o)}X_t$. However, $\mathbb{E}(X_t|\mathcal{F}_t)$ is defined almost surely for any t; thus uncountably many null sets are involved, hence, a priori, $\mathbb{E}(X_t|\mathcal{F}_t)$ is not a well-defined process whereas ${}^{(o)}X$ takes care of this difficulty.

Comment 1.4.1 Let us comment the difficulty here. If X is an integrable random variable, the quantity $\mathbb{E}(X|\mathcal{F}_t)$ is defined a.s., i.e., if $X_t = \mathbb{E}(X|\mathcal{F}_t)$ and $\widetilde{X}_t = \mathbb{E}(X|\mathcal{F}_t)$, then $\mathbb{P}(X_t = \widetilde{X}_t) = 1$. That means that, for any fixed t, there exists a negligeable set Ω_t such that $X_t(\omega) = \widetilde{X}_t(\omega)$ for $\omega \notin \Omega_t$. For processes, we introduce the following definition: the process X is a modification of Y if, for any t, $\mathbb{P}(X_t = Y_t) = 1$. However, one needs a stronger assumption to be able to compare functionals of the processes. The process X is **indistinguishable from** (or a **version** of) Y if $\{\omega : X_t(\omega) = Y_t(\omega), \forall t\}$ is a measurable set and $\mathbb{P}(X_t = Y_t, \forall t) = 1$. If X and Y are modifications of each other and are a.s. continuous, they are indistinguishable.
A difficult, but important result (see Dellacherie [33, p.73]) states: Let X and Y two optional (resp. predictable) processes. If for every finite stopping time (resp. predictable stopping time) τ, $X_\tau = Y_\tau$ a.s., then the processes X and Y are indistinguishable.

Likewise, the **predictable projection** of X is the unique predictable process ${}^{(p)}X$ such that for any $\mathbb{F}$-predictable stopping time τ

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}}) = \mathbb{E}({}^{(p)}X_\tau \mathbb{1}_{\{\tau<\infty\}}) . \tag{1.4.3}$$

As above, this identity reinforces as

$$\mathbb{E}(X_\tau \mathbb{1}_{\{\tau<\infty\}}|\mathcal{F}_{\tau-}) = {}^{(p)}X_\tau \mathbb{1}_{\{\tau<\infty\}} ,$$

for any $\mathbb{F}$-predictable stopping time τ (see Section 1.1 for the definition of $\mathcal{F}_{\tau-}$).

Example 1.4.2 Let $\vartheta_i, i = 1, 2$ be two stopping times such that $\vartheta_1 \leq \vartheta_2$ and Z a bounded r.v.. Let $X = Z\mathbb{1}_{]\!]\vartheta_1,\vartheta_2]\!]}$. Then, ${}^{(o)}X = U\mathbb{1}_{]\!]\vartheta_1,\vartheta_2]\!]}$ ((resp. ${}^{(p)}X = V\mathbb{1}_{]\!]\vartheta_1,\vartheta_2]\!]}$) where U (resp. V) is the right-continuous (resp. left-continuous) version of the martingale $(\mathbb{E}(Z|\mathcal{F}_t), t \geq 0)$.

Let τ and ϑ be two stopping times such that $\vartheta \leq \tau$ and X a positive process. If A is an increasing optional process, then,

$$\mathbb{E}\left(\int_\vartheta^\tau X_t dA_t\right) = \mathbb{E}\left(\int_\vartheta^\tau {}^{(o)}X_t dA_t\right) .$$

If A is an increasing predictable process, then, since $\mathbb{1}_{]\!]\vartheta,\tau]\!]}(t)$ is predictable

$$\mathbb{E}(\int_\vartheta^\tau X_t dA_t) = \mathbb{E}(\int_\vartheta^\tau {}^{(p)}X_t dA_t) .$$

If A is an increasing integrable (hence optional) adapted process, $\mathbb{E}(\int_{[0,\infty[} X_s dA_s) = \mathbb{E}(\int_{[0,\infty[} {}^{(o)}X_s dA_s)$. If A is an increasing integrable predictable process , $\mathbb{E}(\int_{[0,\infty[} X_s dA_s) = \mathbb{E}(\int_{[0,\infty[} {}^{(p)}X_s dA_s)$.

1.4.2 Dual Projections

The notion of interest in this section is that of **dual predictable projection**, which we define as follows:

Proposition 1.4.3 *Let $(A_t, t \geq 0)$ be an integrable increasing process (not necessarily $\mathbb{F}$-adapted). There exists a unique integrable $\mathbb{F}$-predictable increasing process $(A_t^{(p)}, t \geq 0)$, called the dual predictable projection of A such that*

$$\mathbb{E}\left(\int_0^\infty Y_s dA_s\right) = \mathbb{E}\left(\int_0^\infty Y_s dA_s^{(p)}\right)$$

for any positive $\mathbb{F}$-predictable process Y.

In the particular case where $A_t = \int_0^t a_s ds$, one has

$$A_t^{(p)} = \int_0^t {}^{(p)}a_s ds \tag{1.4.4}$$

PROOF: See Dellacherie [34, Chapter V], Dellacherie and Meyer [38, Chapter 6, (73), p. 148], or Protter [101, Chapter 3, Section 5]. The integrability condition of $(A_t^{(p)}, t \geq 0)$ results from the definition, since for $Y = 1$, one obtains $\mathbb{E}(A_{\infty-}^{(p)}) = \mathbb{E}(A_{\infty-})$. □
The dual optional projection is also useful

Proposition 1.4.4 *Let $(A_t,\, t \geq 0)$ be an integrable increasing process (not necessarily $\mathbb{F}$-adapted). There exists a unique integrable $\mathbb{F}$-optional increasing process $(A_t^{(o)}, t \geq 0)$, called the dual optional projection of A such that*

$$\mathbb{E}\left(\int_0^\infty Y_s dA_s\right) = \mathbb{E}\left(\int_0^\infty Y_s dA_s^{(o)}\right)$$

for any positive $\mathbb{F}$-optional process Y.

In the particular case where $A_t = \int_0^t a_s ds$, one has

$$A_t^{(o)} = \int_0^t {}^{(o)}a_s ds \tag{1.4.5}$$

This definition extends to the difference between two integrable increasing processes. The terminology "dual predictable projection" refers to the fact that it is the random measure $d_t A_t(\omega)$ which is relevant when performing that operation:

$$\mathbb{E}\left(\int_0^\infty Y_s dA_s\right) = \mathbb{E}\left(\int_0^\infty Y_s dA_s^{(p)}\right)$$

for any positive $\mathbb{F}$-measurable process Y. Note that the predictable projection of an increasing process is not necessarily increasing, whereas its dual predictable projection is.

If X is bounded and A (not necessarily adapted) has integrable variation, then

$$\mathbb{E}((X \star A^{(p)})_\infty) = \mathbb{E}(({}^{(p)}X \star A)_\infty)\,.$$

This is equivalent to: for $s < t$,

$$\mathbb{E}(A_t - A_s | \mathcal{F}_s) = \mathbb{E}(A_t^{(p)} - A_s^{(p)} | \mathcal{F}_s)\,. \tag{1.4.6}$$

Hence, if A is $\mathbb{F}$-adapted (not necessarily predictable), then $(A_t - A_t^{(p)}, t \geq 0)$ is an $\mathbb{F}$-martingale. In that case, $A^{(p)}$ is also called the predictable compensator of A.

Example 1.4.5 If N is a Poisson process, $N_t^{(p)} = \lambda t$. If X is a Lévy process with Lévy measure ν and f a positive function with compact support which does not contain 0, the predictable compensator of $\sum_{s \leq t} f(\Delta X_s)$ is $t \int f(x)\nu(dx)$

In a general setting, the predictable projection of an increasing process A is a sub-martingale whereas the dual predictable projection is an increasing process. The predictable projection and the dual predictable projection of an increasing process A are equal if and only if ${}^{(p)}A$ is increasing.

Proposition 1.4.6 *If A is increasing, the process ${}^{(o)}A$ is a sub-martingale and $A^{(p)}$ is the predictable increasing process in the Doob-Meyer decomposition of the sub-martingale ${}^{(o)}A$. The process ${}^{(o)}A - A^{(p)}$ is a martingale.*

PROOF: Apply (1.4.1) and (1.4.6). □

Using that terminology, for two martingales X, Y, the predictable covariation process $\langle X, Y\rangle$ is the dual predictable projection of the covariation process $[X, Y]$. The predictable covariation process depends on the filtration.

Example

We now present an example of computation of dual predictable projection. Let $(B_s)_{s\geq 0}$ be an $\mathbb{F}-$Brownian motion starting from 0 and $B_s^{(\nu)} = B_s + \nu s$. Let $\mathbb{G}^{(\nu)}$ be the filtration generated by the process $(|B_s^{(\nu)}|, s \geq 0)$ (which coincides with the one generated by $(B_s^{(\nu)})^2$) (note that $\mathbb{G}^{(\nu)} \subset \mathbb{F}$). We now compute the decomposition of the semi-martingale $(B^{(\nu)})^2$ in the filtration $\mathbb{G}^{(\nu)}$ and the $\mathbb{G}^{(\nu)}$-dual predictable projection of the finite variation process $\int_0^t B_s^{(\nu)} ds$.

Itô's lemma provides us with the decomposition of the process $(B^{(\nu)})^2$ in the filtration $\mathbb{F}$:

$$(B_t^{(\nu)})^2 = 2\int_0^t B_s^{(\nu)} dB_s + 2\nu \int_0^t B_s^{(\nu)} ds + t\,. \tag{1.4.7}$$

To obtain the decomposition in the filtration $\mathbb{G}^{(\nu)}$ we remark that,

$$\mathbb{E}(e^{\nu B_s}|\mathcal{F}_s^{|B|}) = \cosh(\nu B_s)(= \cosh(\nu|B_s|))$$

which leads, thanks to Girsanov's Theorem to the equality:

$$\mathbb{E}(B_s + \nu s|\mathcal{F}_s^{|B|}) = \frac{\mathbb{E}(B_s e^{\nu B_s}|\mathcal{F}_s^{|B|})}{\mathbb{E}(e^{\nu B_s}|\mathcal{F}_s^{|B|})} = B_s \tanh(\nu B_s) = \psi(\nu B_s)/\nu\,,$$

where $\psi(x) = x\tanh(x)$. We now come back to equality (1.4.7). Due to (1.4.4), we have just shown that:

$$\text{The dual predictable projection of } 2\nu\int_0^t B_s^{(\nu)} ds \text{ is } 2\int_0^t ds\,\psi(\nu B_s^{(\nu)})\,. \tag{1.4.8}$$

As a consequence,

$$(B_t^{(\nu)})^2 - 2\int_0^t ds\,\psi(\nu B_s^{(\nu)}) - t$$

is a $\mathbb{G}^{(\nu)}$-martingale with increasing process $4\int_0^t (B_s^{(\nu)})^2 ds$. Hence, there exists a $\mathbb{G}^{(\nu)}$-Brownian motion β such that

$$(B_t + \nu t)^2 = 2\int_0^t |B_s + \nu s| d\beta_s + 2\int_0^t ds\,\psi(\nu(B_s + \nu s)) + t\,. \tag{1.4.9}$$

Compensator of a random time

It will be convenient to introduce the following terminology:

Definition 1.4.7 *If τ is a random time, we call the $\mathbb{F}$-**predictable compensator** associated with τ the $\mathbb{F}$-dual predictable projection $A^{(p)}$ of the increasing process $1\!\!1_{\{\tau\leq t\}}$. This dual predictable projection $A^{(p)}$ satisfies*

$$\mathbb{E}(Y_\tau) = \mathbb{E}\left(\int_0^\infty Y_s dA_s^{(p)}\right) \tag{1.4.10}$$

for any positive, $\mathbb{F}$-predictable process Y.

In case of possible confusion, we shall denote $A^{(p,\tau)}$, or even $A^{(p,\tau,\mathbb{F})}$ this projection.

The process $Z_t = \mathbb{P}(t < \tau|\mathcal{F}_t)$ is the optional projection of $1\!\!1_{]0,\tau[}$ and is a right-continuous supermartingale, the process Z_{t-} is the predictable projection of $1\!\!1_{]0,\tau[}$ (see [35, Chapter XX]).

Proposition 1.4.8 *The Doob-Meyer decomposition of the super-martingale $Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ is*

$$Z_t = \mathbb{E}(A_\infty^{(p)}|\mathcal{F}_t) - A_t^{(p)} = \mu_t - A_t^{(p)}$$

where $\mu_t := \mathbb{E}(A_\infty^{(p)}|\mathcal{F}_t)$ is the martingale part of Z.

Proof: From the definition of the dual predictable projection, for any predictable process Y, one has

$$\mathbb{E}(Y_\tau) = \mathbb{E}\left(\int_0^\infty Y_u dA_u^{(p)}\right).$$

Let t be fixed and $F_t \in \mathcal{F}_t$. Then, the process $Y_u = F_t 1\!\!1_{\{t<u\}}, u \geq 0$ is $\mathbb{F}$-predictable. Then

$$\mathbb{E}(F_t 1\!\!1_{\{t<\tau\}}) = \mathbb{E}(F_t(A_\infty^{(p)} - A_t^{(p)})).$$

It follows that $\mathbb{E}(A_\infty^{(p)}|\mathcal{F}_t) = Z_t + A_t^{(p)}$. Note that μ is a non-negative martingale. □

Proposition 1.4.9 *Let τ be a totally inaccessible stopping time for a filtration $\mathbb{F}$.*
a) The process $H_t = 1\!\!1_{\tau \leq t}$ is a submartingale, and there exists a continuous increasing, $\mathbb{F}$-adapted process $C = (C_t), t \geq 0$ such that $H - C$ is an $\mathbb{F}$-martingale.
b) If the process C is absolutely continuous with respect to Lebesgue measure, then the compensator of τ is AC in any smaller filtration and in particular $F(t) = P(\tau \leq t)$ is an absolutely continuous function.
c) There exists an event $\Gamma \in \mathcal{G}_\tau$ such that τ_Γ has an AC compensator and the compensator of τ_{Γ^c} is not AC

Notation: We shall use frequently the two following conditions :
condition **(A)**: the random time τ avoids the $\mathbb{F}$-stopping times, i.e., $\mathbb{P}(\tau = \vartheta) = 0$ for any $\mathbb{F}$-stopping time ϑ
condition **(C)** : all $\mathbb{F}$-martingales are continuous

Lemma 1.4.10 *Let τ a random time, $A^{(p)}$ be the $\mathbb{F}$-dual predictable projection of the process $H_t := 1\!\!1_{\tau \leq t}$ and let $A^{(o)}$ be the dual optional projection of H.*
*1) Assume condition **(A)**, then $A^{(p)} = A^{(o)}$ and these processes are continuous.*
*2) Under conditions **(C)** and **(A)**, $Z_t := \mathbb{P}(\tau > t|\mathcal{F}_t)$ is continuous.*

Proof: Indeed, if ϑ is a jump time of $A^{(p)}$, it is an $\mathbb{F}$-stopping time, hence is predictable, and

$$\mathbb{E}(A_\vartheta^{(p)} - A_{\vartheta-}^{(p)}) = \mathbb{E}(1\!\!1_{\tau=\vartheta}) = 0;$$

the continuity of $A^{(p)}$ follows.
See Dellacherie and Meyer [38] or Nikeghbali [97].

Lemma 1.4.11 *Let τ be a finite random time such that its associated Azéma's supermartingale Z is continuous. Then τ avoids $\mathbb{F}$-stopping times.*

Proof: See Coculescu and Nikeghbali

Comment 1.4.12 It can be proved that the martingale

$$\mu_t := \mathbb{E}(A_\infty^{(p)}|\mathcal{F}_t) = A_t^{(p)} + Z_t$$

is BMO. We recall that a continuous uniformly integrable martingale M belongs to BMO space if there exists a constant m such that

$$\mathbb{E}(\langle M\rangle_\infty - \langle M\rangle_\tau|\mathcal{F}_\tau) \leq m$$

for any stopping time τ. It can be proved (see, e.g., Dellacherie and Meyer [38, Chapter VII]) that the space BMO is the dual of $\mathbb{H}^1$, the space of martingales such that $\mathbb{E}(\sup_{t\geq 0}|M_t|) < \infty$. Recall that $\mathcal{M}_{loc} = \mathbb{H}^1_{loc}$.

Exercise 1.4.13 Let M a càdlàg martingale. Prove that its predictable projection is M_{t-}. ◁

Exercise 1.4.14 Let X be a measurable process such that $\mathbb{E}(\int_0^t |X_s|ds) < \infty$ and $Y_t = \int_0^t X_s ds$. . Prove that ${}^{(o)}Y_t - \int_0^t {}^{(o)}X_s ds$ is an $\mathbb{F}$-martingale ◁

Exercise 1.4.15 Prove that if X is bounded and Y predictable ${}^{(p)}(YX) = Y\,{}^{(p)}X$ ◁

Exercise 1.4.16 Prove that, more generally than (1.4.8), the dual predictable projection of $\int_0^t f(B_s^{(\nu)})ds$ is $\int_0^t \mathbb{E}(f(B_s^{(\nu)})|\mathcal{G}_s^{(\nu)})ds$ and that

$$\mathbb{E}(f(B_s^{(\nu)})|\mathcal{G}_s^{(\nu)}) = \frac{f(B_s^{(\nu)})e^{\nu B_s^{(\nu)}} + f(-B_s^{(\nu)})e^{-\nu B_s^{(\nu)}}}{2\cosh(\nu B_s^{(\nu)})}.$$

◁

Exercise 1.4.17 Prove that, if $(\alpha_s, s\geq 0)$ is an increasing $\mathbb{F}$-predictable process and X a positive measurable process, then

$$\left(\int_0^\cdot X_s d\alpha_s\right)^{(p)} = \int_0^\cdot {}^{(p)}X_s d\alpha_s$$

In particular

$$\left(\int_0^\cdot X_s ds\right)^{(p)} = \int_0^\cdot {}^{(p)}X_s ds$$

◁

Exercise 1.4.18 Give an example of random time τ where $A^{(p)}$ and $A^{(o)}$ are different. ◁

✓TO COMPLETE

1.5 Some Important Exercices

Exercise 1.5.1 Let B be a Brownian motion, $\mathbb{F}$ its natural filtration and $B_t^* = \sup_{s\leq t} B_s$. Prove that, for $t < 1$,

$$\mathbb{E}(f(B_1^*)|\mathcal{F}_t) = F(1-t, B_t, B_t^*)$$

with

$$F(s,a,b) = \sqrt{\frac{2}{\pi s}}\left(f(b)\int_0^{b-a} e^{-u^2/(2s)}du + \int_b^\infty f(u)\exp\left(-\frac{(u-a)^2}{2s}\right)du\right).$$

`Hint:` Note that

$$\sup_{s\leq 1} B_s = \sup_{s\leq t} B_s \vee \sup_{t\leq s\leq 1} B_s = \sup_{s\leq t} B_s \vee (\widehat{B}^*_{1-t} + B_t)$$

where $\widehat{B}^*_s = \sup_{u\leq s}\widehat{B}_u$ for $\widehat{B}_u = B_{u+t} - B_t$. ◁

Exercise 1.5.2 Let φ be a C^1 function, B a Brownian motion and $B^*_t = \sup_{s\leq t} B_s$. Prove that the process

$$\varphi(B^*_t) - (B^*_t - B_t)\varphi'(B^*_t)$$

is a local martingale. $\triangleleft$

Exercise 1.5.3 A Useful Lemma: Doob's Maximal Identity.
Let M be a positive continuous martingale such that $M_0 = x$.

(i) Prove that if $\lim_{t\to\infty} M_t = 0$, then

$$\mathbb{P}(\sup M_t > a) = \left(\frac{x}{a}\right) \wedge 1 \tag{1.5.1}$$

and $\sup M_t \overset{\text{law}}{=} \dfrac{x}{U}$ where U is a random variable with a uniform law on $[0,1]$.

(ii) Conversely, if $\sup M_t \overset{\text{law}}{=} \dfrac{x}{U}$, show that $M_\infty = 0$.

`Hint:` Apply Doob's optional sampling theorem to $T_a \wedge t$ and prove, passing to the limit when t goes to infinity, that

$$x = \mathbb{E}(M_{T_a}) = a\mathbb{P}(T_a < \infty) = a\mathbb{P}(\sup M_t \geq a)\,.$$

$\triangleleft$

Exercise 1.5.4 Let $\mathbb{F} \subset \mathbb{G}$ and $d\mathbb{Q}|_{\mathcal{G}_t} = L_t dP|_{\mathcal{G}_t}$, where L is continuous. Prove that $d\mathbb{Q}|_{\mathcal{F}_t} = \ell_t d\mathbb{P}|_{\mathcal{F}_t}$ where $\ell_t = \mathbb{E}(L_t|\mathcal{F}_t)$. Prove that any $(\mathbb{G},\mathbb{Q})$-martingale can be written as $M^{\mathbb{P}}_t - \int_0^t \frac{d\langle M^{\mathbb{P}}, L\rangle_s}{L_s}$ where $M^{\mathbb{P}}$ is a $(\mathbb{G},\mathbb{P})$-martingale. $\triangleleft$

Exercise 1.5.5 Prove that, for any (bounded) process a (not necessarily adapted)

$$M_t := \mathbb{E}(\int_0^t a_u du|\mathcal{F}_t) - \int_0^t \mathbb{E}(a_u|\mathcal{F}_u)du$$

is an $\mathbb{F}$-martingale. Extend the result to the case $\int_0^{\cdot} X_s d\alpha_s$ where $(\alpha_s, s \geq 0)$ is an increasing predictable process and X a positive measurable process. $\triangleleft$

`Hint:` Compute $\mathbb{E}(M_t - M_s|\mathcal{F}_s) = \mathbb{E}(\int_s^t a_u du - \int_s^t \mathbb{E}(a_u|\mathcal{F}_u)du|\mathcal{F}_s)$.

Chapter 2

Compensators, Single Default

The $\mathbb{F}$-compensator of a càdlàg $\mathbb{F}$-submartingale X is the càdlàg increasing and $\mathbb{F}$-predictable process A such that $X - A$ is an $\mathbb{F}$-martingale. From Doob-Meyer decomposition, the compensator exists if X is of class (D). Of course, the value of the compensator depends on the underlying filtration.

An important example is a Poisson process N, with constant intensity λ. In that case, the increasing process N (a sub-martingale) admits $A_t = \lambda t$ as compensator (in its own filtration).

In this chapter, we shall study in more details compensators of increasing processes (which are obviously submartingales), in particular compensators of $1\!\!1_{\tau \leq t}$ for a positive random variable τ. Let us note that, if $\mathbb{F}$ is a Brownian filtration and τ an $\mathbb{F}$-stopping time (or more generally, if τ is an $\mathbb{F}$-predictable stopping time), the $\mathbb{F}$-compensator of $1\!\!1_{\tau \leq t}$ is $1\!\!1_{\tau \leq t}$.

2.1 A Toy Model

Let τ be a random time (a non-negative random variable) on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We denote by $(H_t, t \geq 0)$ the right-continuous increasing process $H_t = 1\!\!1_{\{\tau \leq t\}}$ and by $\mathbb{H} = (\mathcal{H}_t, t \geq 0)$ its natural filtration (which is right-continuous) completed with the negligible sets (See Dellacherie [33]). The filtration $\mathbb{H}$ is the smallest filtration which satisfies usual hypothesis, which makes τ a stopping time. The σ-algebra $\mathcal{H}_t$ is generated by the sets $\{\tau \leq s\}$ for $s \leq t$ (note that the set $\{\tau > t\}$ is an atom of $\mathcal{H}_t$). A key point is that any integrable $\mathcal{H}_t$-measurable r.v. K is of the form $K = g(\tau)1\!\!1_{\{\tau \leq t\}} + h(t)1\!\!1_{\{t < \tau\}}$ where g, h are Borel functions. We denote by F the (right-continuous) cumulative distribution function of τ, defined as $F(t) = \mathbb{P}(\tau \leq t)$, and by G the survival function $G(t) = 1 - F(t)$.
We first give some elementary tools to compute the conditional expectation w.r.t. $\mathcal{H}_t$, as presented in Brémaud [24], Dellacherie [33, 34], Elliott [43]. Note that if the cumulative distribution function of τ is continuous, then, τ is an $\mathbb{H}$-totally inaccessible stopping time. (See Dellacherie and Meyer [38, Chapter IV, p.107].)

2.1.1 Key Lemma

Lemma 2.1.1 *If X is any integrable, $\mathcal{A}$-measurable r.v., one has*

$$\mathbb{E}(X|\mathcal{H}_s)1\!\!1_{\{s<\tau\}} = 1\!\!1_{\{s<\tau\}} \frac{\mathbb{E}(X 1\!\!1_{\{s<\tau\}})}{\mathbb{P}(s < \tau)} \,. \tag{2.1.1}$$

PROOF: The r.v. $\mathbb{E}(X|\mathcal{H}_s)$ is $\mathcal{H}_s$-measurable. Therefore, it can be written in the form $\mathbb{E}(X|\mathcal{H}_s) = g(\tau)1\!\!1_{\{s \geq \tau\}} + h(s)1\!\!1_{\{s<\tau\}}$ for some functions g, h. By multiplying both members by $1\!\!1_{\{s<\tau\}}$, and taking the expectation, we obtain, using the fact that $\{s < \tau\} \in \mathcal{H}_s$,

$$\begin{aligned} \mathbb{E}[1\!\!1_{\{s<\tau\}}\mathbb{E}(X|\mathcal{H}_s)] &= \mathbb{E}[\mathbb{E}(1\!\!1_{\{s<\tau\}}X|\mathcal{H}_s)] = \mathbb{E}[1\!\!1_{\{s<\tau\}}X] \\ &= \mathbb{E}(h(s)1\!\!1_{\{s<\tau\}}) = h(s)\mathbb{P}(s<\tau)\,. \end{aligned}$$

Hence, if $\mathbb{P}(s < \tau) \neq 0$, $h(s) = \dfrac{\mathbb{E}(X1\!\!1_{\{s<\tau\}})}{\mathbb{P}(s<\tau)}$ gives the desired result. If, for some s, one has $\mathbb{P}(s < \tau) = 0$, then $\{\tau > s\}$ is a negligeable set and $1\!\!1_{s<\tau} = 0$ a.s. Then, in the right-hand side of (2.1.1), we set $\frac{0}{0} = 0$. □

Exercise 2.1.2 Assume that Y is $\mathcal{H}_\infty$-measurable, so that $Y = h(\tau)$ for some Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}$ and that $F(t) < 1$ for $t > 0$. Let $\Gamma(t) = -\ln(\mathbb{P}(\tau > t))$ be the hazard function of τ and assume that Γ is continuous. Prove that

$$\mathbb{E}(Y|\mathcal{H}_t) = 1\!\!1_{\{\tau\le t\}}h(\tau) + 1\!\!1_{\{t<\tau\}}\int_t^\infty h(u)e^{\Gamma(t)-\Gamma(u)}\,d\Gamma(u). \tag{2.1.2}$$

◁

2.1.2 Some Martingales

In all this section, we assume that F is continuous.

Proposition 2.1.3 *Assuming that F is continuous and $F(t) < 1, \forall t$, the process $(M_t, t \ge 0)$ defined as*

$$M_t = H_t - \int_0^{\tau\wedge t}\frac{dF(s)}{1-F(s)} = H_t - \int_0^t (1-H_{s-})\frac{dF(s)}{1-F(s)} = H_t + \int_0^t (1-H_{s-})\frac{dG(s)}{G(s)}$$

is a $\mathbb{H}$-martingale.

Proof: Let $s < t$. Then:

$$\mathbb{E}(H_t - H_s|\mathcal{H}_s) = 1\!\!1_{\{s<\tau\}}\mathbb{E}(1\!\!1_{\{s<\tau\le t\}}|\mathcal{H}_s) = 1\!\!1_{\{s<\tau\}}\frac{F(t)-F(s)}{1-F(s)}\,, \tag{2.1.3}$$

which follows from (2.1.1) with $X = 1\!\!1_{\{\tau\le t\}}$.
On the other hand, the quantity

$$C := \mathbb{E}\left[\int_s^t (1-H_{u-})\frac{dF(u)}{1-F(u)}\,|\,\mathcal{H}_s\right],$$

is equal to

$$\begin{aligned} C &= \int_s^t \frac{dF(u)}{1-F(u)}\mathbb{E}\left[1\!\!1_{\{\tau>u\}}|\mathcal{H}_s\right] \\ &= 1\!\!1_{\{\tau>s\}}\int_s^t \frac{dF(u)}{1-F(u)}\left(1-\frac{F(u)-F(s)}{1-F(s)}\right) = 1\!\!1_{\{\tau>s\}}\int_s^t\frac{dF(u)}{1-F(s)} \\ &= 1\!\!1_{\{\tau>s\}}\frac{F(t)-F(s)}{1-F(s)} \end{aligned}$$

which, from (2.1.3) proves the desired result. □

The (continuous increasing) function

$$\Gamma(t) := \int_0^t \frac{dF(s)}{1-F(s)} = -\ln(1-F(t)) = -\ln(G(t))$$

is called the **hazard function** of τ. Note, for future use, that $dF(t) = G(t)d\Gamma(t) = e^{-\Gamma(t)}d\Gamma(t)$.
From Proposition 2.1.3, we obtain that the process $M_t := H_t - \Gamma(t \wedge \tau)$ is an $\mathbb{H}$ martingale, hence the Doob-Meyer decomposition of the submartingale H is $H_t = M_t + \Gamma(t \wedge \tau)$. The (predictable) process $A_t = \Gamma_{t\wedge\tau}$ is called the **compensator** of H.
Moreover, if F is differentiable, the process

$$M_t = H_t - \int_0^{\tau\wedge t} \gamma(s)ds = H_t - \int_0^t \gamma(s)(1 - H_s)ds$$

is a martingale, where $\gamma(s) = \dfrac{f(s)}{1 - F(s)}$ is a deterministic non-negative function, called **the intensity of** τ.

Proposition 2.1.4 *Assume that F is a continuous function. For any (bounded) Borel measurable function $h : \mathbb{R}_+ \to \mathbb{R}$, the process*

$$M_t^h = 1\!\!1_{\{\tau\le t\}} h(\tau) - \int_0^{t\wedge\tau} h(u)\, d\Gamma(u) \tag{2.1.4}$$

is a $\mathbb{H}$-martingale.

PROOF: On the one hand, for $s < t$,

$$\begin{aligned}
\mathbb{E}\big(h(\tau)1\!\!1_{\{s<\tau\le t\}} \,|\, \mathcal{H}_s\big) &= 1\!\!1_{\{s<\tau\}} \frac{1}{\mathbb{P}(s<\tau)} \mathbb{E}(h(\tau)1\!\!1_{s<\tau\le t}) = 1\!\!1_{\{s<\tau\}} e^{\Gamma(s)} \int_s^t h(u)dF(u) \\
&= 1\!\!1_{\{s<\tau\}} e^{\Gamma(s)} \int_s^t h(u)e^{-\Gamma(u)}\, d\Gamma(u).
\end{aligned}$$

On the other hand, we get

$$J := \mathbb{E}\Big(\int_{s\wedge\tau}^{t\wedge\tau} h(u)\, d\Gamma(u) | \mathcal{H}_s\Big) = \mathbb{E}\big(\tilde h(\tau)1\!\!1_{\{s<\tau\le t\}} + \tilde h(t) 1\!\!1_{\{\tau>t\}} \,|\, \mathcal{H}_s\big)$$

where, for fixed s, we set $\tilde h(t) = \int_s^t h(u)\, d\Gamma(u)$. Consequently,

$$J = 1\!\!1_{\{s<\tau\}} e^{\Gamma(s)} \Big(\int_s^t \tilde h(u)e^{-\Gamma(u)}\, d\Gamma(u) + e^{-\Gamma(t)}\tilde h(t)\Big) =: 1\!\!1_{\{s<\tau\}} e^{\Gamma(s)} \widetilde J.$$

To conclude the proof, it is enough to observe that Fubini's theorem yields

$$\begin{aligned}
\widetilde J &= \int_s^t d\Gamma(u) e^{-\Gamma(u)} \int_s^u h(v)\, d\Gamma(v) + e^{-\Gamma(t)}\tilde h(t) \\
&= \int_s^t d\Gamma(u) h(u) \int_u^t e^{-\Gamma(v)}\, d\Gamma(v) \,+ e^{-\Gamma(t)} \int_s^t h(u)\, d\Gamma(u) \\
&= \int_s^t h(u) e^{-\Gamma(u)}\, d\Gamma(u),
\end{aligned}$$

as expected. □

Example 2.1.5 In the case where N is an inhomogeneous Poisson process with deterministic intensity λ and τ is the first time when N jumps, let $H_t = N_{t\wedge\tau}$. It is well known that $N_t - \int_0^t \lambda(s)ds$ is a martingale (indeed, N can be viewed as a standard Poisson process $\widetilde N$ of intensity 1, changed of time; $N_t = \widetilde N_{\Lambda(t)}$ with $\Lambda_t = \int_0^t \lambda(s)ds$). Therefore, the process stopped at time τ is also a martingale, i.e., $H_t - \int_0^{t\wedge\tau} \lambda(s)ds$ is a martingale.

Exercise 2.1.6 Let B be a Brownian motion and $\tau = \inf\{t \; B_t = a\}$. Find the $\mathbb{F}^B$ compensator of τ. Find the $\mathbb{F}^0$ compensator of τ, when $\mathbb{F}^0$ is the trivial filtration. ◁

Exercise 2.1.7 Write M^h as an integral w.r.t. M. ◁

Exercise 2.1.8 a) Prove that the process $L_t := 1\!\!1_{\{\tau>t\}} \exp\left(\int_0^t \gamma(s)ds\right)$ is an $\mathbb{H}$-martingale and

$$L_t = 1 - \int_{]0,t]} L_{u-} dM_u \tag{2.1.5}$$

In particular, for $t < T$,

$$\mathbb{E}(1\!\!1_{\{\tau>T\}}|\mathcal{H}_t) = 1\!\!1_{\{\tau>t\}} \exp\left(-\int_t^T \gamma(s)ds\right).$$

b) Let $d\mathbb{Q}_{|\mathcal{H}_t} = L_t d\mathbb{P}_{|\mathcal{H}_t}$. Prove that $\mathbb{Q}(\tau \leq t) = 0$. ◁

Exercise 2.1.9 a) Let F be continuous and $h : \mathbb{R}_+ \to \mathbb{R}$ be a (bounded) Borel measurable function. Prove that the process

$$Y_t := \exp\left(1\!\!1_{\{\tau\leq t\}} h(\tau)\right) - \int_0^{t\wedge\tau} (e^{h(u)} - 1)\, d\Gamma(u) \tag{2.1.6}$$

is a $\mathbb{H}$-martingale. Write Y as a Doléans-Dade martingale, i.e., find φ such that

$$dY_t = Y_{t-}\varphi_t dM_t$$

◁

Exercise 2.1.10 Assume that Γ is a continuous function. Let $h : \mathbb{R}_+ \to \mathbb{R}$ be a non-negative Borel measurable function such that the random variable $h(\tau)$ is integrable. Prove that the process

$$Y_t := (1 + 1\!\!1_{\tau\leq t} h(\tau)) \exp\Big(-\int_0^{t\wedge\tau} h(u)\, d\Gamma(u)\Big). \tag{2.1.7}$$

is an $\mathbb{H}$-martingale. Write Y as a Doléans-Dade martingale, i.e., find φ such that

$$dY_t = Y_{t-}\varphi_t dM_t\,.$$

◁

Exercise 2.1.11 In this exercise, F is only continuous on right, and $F(t-)$ is the left limit at point t. Prove that the process $(M_t, t \geq 0)$ defined as

$$M_t = H_t - \int_0^{\tau\wedge t} \frac{dF(s)}{1 - F(s-)} = H_t - \int_0^t (1 - H_s)\frac{dF(s)}{1 - F(s-)}$$

is an $\mathbb{H}$-martingale. ◁

2.1.3 Stopping times

Proposition 2.1.12 *A random variable ϑ is an $\mathbb{H}$-stopping time if and only if there exists a constant $s \in [0, \infty]$ such that*

$\vartheta \geq \tau$ a.s. on the set $\{\tau \leq s\}$
$\vartheta = s$ a.s. on the set $\{\tau > s\}$.

Proof: Since τ is an $\mathbb{H}$-stopping time and $\mathcal{H}_\tau = \mathcal{H}_\infty$, the two conditions are obviously sufficient. Conversely, let ϑ be a stopping time.

2.1.4 Several defaults

We present here a toy model with two random time, to underline the rôle of the filtration

We consider the case where the sources of randomness are the occurrence of two random times τ_1 and τ_2 (finite positive random variables).

We denote by $\mathbb{H}^1$ the filtration generated by the process $(H_t^1 := 1\!\!1_{\tau_1 \leq t})$, by $\mathbb{H}^2$ the filtration generated by the process $(H_t^2 := 1\!\!1_{\tau_2 \leq t})$ and by $\mathbb{G}$ the filtration generated by both processes $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$.

We denote by $G(t,s) = \mathbb{P}(\tau_1 > t, \tau_2 > s)$ the survival probability of the pair (τ_1, τ_2) assumed to be strictly positive and continuously differentiable in both variables. Note that $G(t,0) = \mathbb{P}(\tau_1 > t)$ is the survival probability of τ_1.

We introduce the fundamental martingales

$$\begin{aligned} M_t^1 &:= H_t^1 - \lambda(t \wedge \tau_1) = H_t - \int_0^t (1 - H_s)\lambda ds, \text{ a } (\mathbb{P}, \mathbb{F}) \text{ martingale} \\ M_t^2 &:= H_t^1 - \int_0^t (1 - H_s)\lambda_s^2 ds, \text{ a } (\mathbb{P}, \mathbb{G}) \text{ martingale} \end{aligned}$$

where

$$\lambda_t^2 = 1\!\!1_{t \leq \tau_2} \frac{-\partial_1 G(t,t)}{G(t,t)} + 1\!\!1_{\tau_2 < t} \frac{\partial_{12} G(t, \tau_2)}{-\partial_1 G(t, \tau_2)}$$

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2.2 General case

In a credit risk setting, the random variable τ represents the time when a default occurs. In the literature, models for default times are often based on a threshold: the default occurs when some driving process X reaches a given barrier. Based on this observation, we consider the random time on $I\!\!R_+$ in a general threshold model. Let X be a stochastic process and Θ be a barrier which we shall make precise later. Define the random time as the first passage time

$$\tau := \inf\{t : X_t \geq \Theta\}.$$

In classical structural models, a reference filtration $\mathbb{F}$ is given, the process X is an $\mathbb{F}$-adapted process associated with the value of a firm and the barrier Θ is a constant. So, τ is an $\mathbb{F}$-stopping time. If τ is a predictable stopping time (e.g., if $\mathbb{F}$ is a Brownian filtration), the compensator of $H_t = 1\!\!1_{\tau \leq t}$ is H_t. The goal is then to compute the conditional law of the default $P(\tau > \theta | \mathcal{F}_t^X)$, for $\theta > t$

In reduced form approach (say, if τ is not the first time where a process reaches a constant barrier), we shall deal with two kinds of information: some information denoted as $(\mathcal{F}_t, t \geq 0)$ and the information from the default time, i.e. the knowledge of the time where the default occurred in the past, it the default has appeared. More precisely, this information is modeled by the filtration $\mathbb{H}$ generated by the default process H (completed with negligeable sets).

At the intuitive level, $\mathbb{F}$ is generated by prices of some assets, or by other economic factors (e.g., interest rates). This filtration can also be a subfiltration of the prices. The case where $\mathbb{F}$ is the trivial filtration is exactly what we have studied in the toy model. Though in typical examples $\mathbb{F}$ is chosen to be the Brownian filtration, most theoretical results do not rely on such a specification of the filtration $\mathbb{F}$.

We denote by $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ the enlarged filtration which is the smallest right-continuous filtration which contains $\mathbb{F}$, making τ a stopping time.

2.2.1 Key Lemma

It is straightforward to establish that any $\mathcal{G}_t$-measurable random variable is equal, on the set $\{\tau > t\}$, to an $\mathcal{F}_t$-measurable random variable. Indeed, $\mathcal{G}_t$-measurable random variables are generated by $x_t(g(\tau)\mathbb{1}_{\tau\le t} + h(t)\mathbb{1}_{t<\tau}$, where x_t is $\mathcal{F}_t$ measurable and g, h are Borel functions. In particular, if Y is a $\mathbb{G}$-adapted process, there exists an $\mathbb{F}$-adapted process $Y^{\mathbb{F}}$, called the predefault-value of Y, such that $\mathbb{1}_{\{t<\tau\}}Y_t = \mathbb{1}_{\{t<\tau\}}Y_t^{\mathbb{F}}$. Under the standing assumption that $P(\tau > t \,|\, \mathcal{F}_t) > 0$ for $t \in \mathbb{R}_+$, the uniqueness of pre-default values follows from [35, p.186]. Moreover, if Y is $\mathbb{G}$-predictable its pre-default value $Y^{\mathbb{F}}$ coincide up to τ *included* (see [35, p.186]), namely,

$$\mathbb{1}_{\{t\le\tau\}}Y_t = \mathbb{1}_{\{t\le\tau\}}Y_t^{\mathbb{F}}.$$

If Y is $\mathbb{G}$-adapted, it is standard to check that $Y \ge 0$ implies $Y^{\mathbb{F}} \ge 0$.

We denote by $F_t = \mathbb{P}(\tau \le t|\mathcal{F}_t)$ the conditional cumulative probability of τ given the information $\mathcal{F}_t$ and we set[1] $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = 1 - F_t$. We assume $G_t > 0$, for $t > 0$.

Lemma 2.2.1 ***Key Lemma 1.*** *Let X be an $\mathcal{F}_T$-measurable integrable r.v. Then, for $t \le T$*

$$\mathbb{E}(X\mathbb{1}_{T<\tau}|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}\frac{\mathbb{E}(X\mathbb{1}_{\{\tau>T\}}|\mathcal{F}_t)}{\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}}\frac{1}{G_t}\mathbb{E}(XG_T|\mathcal{F}_t)\,. \tag{2.2.1}$$

PROOF: Note that

$$\mathbb{1}_{\{\tau>t\}}\mathbb{E}(X|\mathcal{G}_t) = \mathbb{1}_{\{\tau>t\}}x_t$$

where x_t is $\mathcal{F}_t$-measurable, and taking conditional expectation w.r.t. $\mathcal{F}_t$ of both members, we deduce

$$x_t = \frac{\mathbb{E}(X\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)}{\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)} = \mathbb{1}_{\{\tau>t\}}\frac{1}{G_t}\mathbb{E}(XG_T|\mathcal{F}_t)\,.$$

□

Note, for future use, that the process G is not necessarily decreasing.

Lemma 2.2.2 ***Key lemma 2.*** *Let h be an $\mathbb{F}$-predictable process. Then, for $t < T$,*

$$\mathbb{E}(h_\tau\mathbb{1}_{\tau<T}|\mathcal{G}_t) = h_\tau\mathbb{1}_{\{\tau<t\}} + \mathbb{1}_{\{\tau>t\}}\frac{1}{G_t}\mathbb{E}(\int_t^T h_u dF_u|\mathcal{F}_t) \tag{2.2.2}$$

PROOF: In a first step, the result is established for processes h of the form $h_t = \mathbb{1}_{]u,v]}(t)K_u$ where $K_u \in \mathcal{F}_u$. In that case, for $t < u < v < T$, applying the key lemma

$$\mathbb{E}(h_\tau\mathbb{1}_{\tau<T}|\mathcal{G}_t) \;=\; \mathbb{E}(K_u\mathbb{1}_{u<\tau<v}|\mathcal{G}_t) = \mathbb{1}_{t<\tau}\frac{1}{G_t}\mathbb{E}(K_u\mathbb{1}_{u<\tau<v}|\mathcal{F}_t)$$

It remains to note that

$$\begin{aligned}\mathbb{E}(K_u\mathbb{1}_{u<\tau<v}|\mathcal{F}_t) &= \mathbb{E}(K_u\mathbb{1}_{\tau<v}|\mathcal{F}_t) - \mathbb{E}(K_u\mathbb{1}_{\tau<u}|\mathcal{F}_t)\\ &= \mathbb{E}(K_u(1-F_v)|\mathcal{F}_t) - \mathbb{E}(K_u(1-F_u)|\mathcal{F}_t) = \mathbb{E}(\int_t^T h_r dF_r|\mathcal{F}_t)\end{aligned}$$

The result follows by approximation. □

As we shall see, this elementary result will allow us to compute the value of credit derivatives.

Comment 2.2.3 It can be useful to understand the meaning of the lemma in the case where, as in the structural model, the default time is an $\mathbb{F}$-stopping time.
We are not interested in this lemma with $\mathbb{G}$-predictable processes, mainly because any $\mathbb{G}$-predictable process is equal, on $\{t \le \tau\}$ to an $\mathbb{F}$-predictable process.

[1] Latter on, we shall denote frequently by Z this quantity, as it is done in the literature on enlargement of filtration.

2.2.2 Martingales

Proposition 2.2.4 *The process $(F_t, t \geq 0)$ is an $\mathbb{F}$-submartingale. The process G is an $\mathbb{F}$-supermartingale.*

PROOF: From definition, and from the increasing property of the process H, for $s < t$:

$$\mathbb{E}(F_t|\mathcal{F}_s) = \mathbb{E}\left(\mathbb{E}(H_t|\mathcal{F}_t)\,|\mathcal{F}_s\right) = \mathbb{E}\left(H_t|\mathcal{F}_s\right) \geq \mathbb{E}\left(H_s|\mathcal{F}_s\right) = F_s\,.$$

□

Proposition 2.2.5 *(i) The process $L_t = (1 - H_t)/G_t$ is a $\mathbb{G}$ -martingale.*

(ii) If X is an $\mathbb{F}$-martingale, XL is a $\mathbb{G}$ -martingale.

(iii) If the process G is decreasing and continuous, the process $M_t = H_t - \Gamma(t \wedge \tau)$ is a $\mathbb{G}$ -martingale where $\Gamma = -\ln G$.

PROOF: (i) From the key lemma, for $t > s$

$$\mathbb{E}(L_t|\mathcal{G}_s) = \mathbb{E}(\mathbb{1}_{\{\tau>t\}}\frac{1}{G_t}|\mathcal{G}_s) = \mathbb{1}_{\{\tau>s\}}\frac{1}{G_s}\mathbb{E}(\mathbb{1}_{\{\tau>t\}}\frac{1}{G_t}|\mathcal{F}_s) = \mathbb{1}_{\{\tau>s\}}\frac{1}{G_s}\mathbb{E}(\frac{1}{G_t}G_t|\mathcal{F}_s) = \mathbb{1}_{\{\tau>s\}}\frac{1}{G_s} = L_s$$

(ii) From the key lemma,

$$\begin{aligned}
\mathbb{E}(L_tX_t|\mathcal{G}_s) &= \mathbb{E}(\mathbb{1}_{\{\tau>t\}}L_tX_t|\mathcal{G}_s)\\
&= \mathbb{1}_{\{\tau>s\}}\frac{1}{G_s}\mathbb{E}(\mathbb{1}_{\{\tau>t\}}G_tX_t|\mathcal{F}_s)\\
&= \mathbb{1}_{\{\tau>s\}}\frac{1}{G_s}\mathbb{E}(\mathbb{E}(\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_t)G_tX_t|\mathcal{F}_s) = L_s\mathbb{E}(X_t|\mathcal{F}_s) = L_sX_s.
\end{aligned}$$

(iii) From integration by parts formula (H is a finite variation process, and Γ an increasing continuous process):

$$dL_t = (1 - H_t)e^{\Gamma_t}d\Gamma_t - e^{\Gamma_t}dH_t$$

and the process $M_t = H_t - \Gamma(t \wedge \tau)$ can be written

$$M_t \equiv \int_{]0,t]} dH_u - \int_{]0,t]} (1 - H_u)d\Gamma_u = -\int_{]0,t]} e^{-\Gamma_u}dL_u$$

and is a $\mathbb{G}$-local martingale since L is $\mathbb{G}$-martingale. (It can be noted that, if Γ is not increasing, the differential of e^{Γ} is more complicated.) □

Comment 2.2.6 Assertion (ii) seems to be related with a change of probability. It is important to note that here, one changes the filtration, not the probability measure. Moreover, setting $d\mathbb{Q}^* = Ld\mathbb{P}$ does not define a probability $\mathbb{Q}$ **equivalent** to $\mathbb{P}$, since the positive martingale L vanishes. The probability $\mathbb{Q}^*$ would be only absolutely continuous w.r.t. $\mathbb{P}$. See Collin-Dufresne and Hugonnier [30].

Lemma 2.2.7 *The process G is a super-martingale.*

As a supermartingale, G admits a Doob-Meyer decomposition

$$G_t = \mu_t - A^p_t \tag{2.2.3}$$

where μ is a martingale and A^p is a predictable increasing process.

Proposition 2.2.8 *Assume that $G_t > 0$. Let A^p be defined in (2.2.3). The process*

$$M_t = H_t - \int_0^{t\wedge\tau} \frac{dA^p_u}{G_{u-}}$$

is a $\mathbb{G}$-martingale.

PROOF: We give the proof in the case where G is continuous in two steps. In the proof $A = A^p$. In a first step, we prove that, for $s < t$

$$\mathbb{E}(H_t|\mathcal{G}_s) = H_s + 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

Indeed,

$$\begin{aligned}
\mathbb{E}(H_t|\mathcal{G}_s) &= 1 - \mathbb{P}(t<\tau|\mathcal{G}_s) = 1 - 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}(G_t|\mathcal{F}_s) = 1 - 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}(\mu_t - A_t|\mathcal{F}_s)\\
&= 1 - 1\!\!1_{s<\tau}\frac{1}{G_s}(\mu_s - A_s - \mathbb{E}(A_t - A_s|\mathcal{F}_s)) = 1 - 1\!\!1_{s<\tau}\frac{1}{G_s}(\mu_s - \mathbb{E}(A_t - A_s|\mathcal{F}_s))\\
&= 1\!\!1_{\tau\le s} + 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)\,.
\end{aligned}$$

In a second step, we prove that, setting, for any v, $K_v = \int_0^v (1-H_s)\frac{dA_s}{G_s}$,

$$\mathbb{E}(K_{t\wedge\tau}|\mathcal{G}_s) = K_{s\wedge\tau} + 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}(A_t - A_s|\mathcal{F}_s)$$

Indeed, from the key formula, for fixed t and $h_u = K_{t\wedge u}$

$$\begin{aligned}
\mathbb{E}(K_{t\wedge\tau}|\mathcal{G}_s) &= K_{t\wedge\tau}1\!\!1_{\tau\le s} + 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(\int_s^\infty K_{t\wedge u}dF_u|\mathcal{F}_s\right)\\
&= K_\tau 1\!\!1_{\tau\le s} + 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(\int_s^t K_u dF_u + \int_t^\infty K_t dF_u|\mathcal{F}_s\right)\\
&= K_{s\wedge\tau}1\!\!1_{\tau\le s} + 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(\int_s^t K_u dF_u + K_t G_t|\mathcal{F}_s\right)
\end{aligned}$$

We now use IP formula, using the fact that K has finite variation and is continuous

$$d(K_t(1-F_t)) = -K_t dF_t + (1-F_t)dK_t = -K_t dF_t + dA_t$$

hence

$$\int_s^t K_u dF_u + K_t(1-F_t) = -K_t(1-F_t) + K_s(1-F_s) + A_t - A_s + K_t(1-F_t) = K_s(1-F_s) + A_t - A_s\,.$$

It follows that

$$\begin{aligned}
\mathbb{E}(K_{t\wedge\tau}|\mathcal{G}_s) &= K_{s\wedge\tau}1\!\!1_{\tau\le s} + 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(K_s G_s + A_t - A_s|\mathcal{F}_s\right)\\
&= K_{s\wedge\tau} + 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(A_t - A_s|\mathcal{F}_s\right)\,.
\end{aligned}$$

Assuming that A is absolutely continuous w.r.t. the Lebesgue measure and denoting by a its derivative, we have proved the existence of a $\mathbb{F}$-adapted process λ, called the intensity rate such that the process

$$H_t - \int_0^{t\wedge\tau}\lambda_u du = H_t - \int_0^t (1-H_u)\lambda_u du$$

is a $\mathbb{G}$-martingale. More precisely, $\lambda_s = \frac{a_s}{1-F_s}$.

Lemma 2.2.9 *The process λ satisfies*

$$\lambda_t = \lim_{h \to 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t+h | \mathcal{F}_t)}{\mathbb{P}(t < \tau | \mathcal{F}_t)} .$$

PROOF: The martingale property of M implies that

$$\mathbb{E}(\mathbb{1}_{t<\tau<t+h} | \mathcal{G}_t) = \int_t^{t+h} \mathbb{E}((1 - H_s)\lambda_s | \mathcal{G}_t) ds$$

It follows that, on $\{t < \tau\}$

$$\lambda_t = \frac{1}{h} \lim \mathbb{P}(t < \tau < t+h | \mathcal{G}_t) = \lim_{h \to 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t+h | \mathcal{F}_t)}{\mathbb{P}(t < \tau | \mathcal{F}_t)} .$$

□

Comment 2.2.10 We assume G continuous. We recall that the Doob-Meyer decomposition of G is denoted $G_t = \mu_t - A_t$. From $L_t = (1 - H_t)(G_t)^{-1}$, one obtains

$$dL_t = -(1 - H_{t-}) \frac{1}{G_t^2} (d\mu_t - dA_t) + \frac{1}{G_t^3} d\langle\mu\rangle_t - \frac{1}{G_t} dH_t$$

it follows that

$$dL_t - \frac{1}{G_t} dM_t = -(1 - H_t) \frac{1}{G_t^2} \left(d\mu_t - \frac{1}{G_t} d\langle\mu\rangle_t \right)$$

hence, due to the $\mathbb{G}$-martingale property of L, the quantity $(1 - H_t)\frac{1}{G_t^2} \left(d\mu_t - \frac{1}{G_t} d\langle\mu\rangle_t \right)$ corresponds to a $\mathbb{G}$-local martingale.

2.3 Cox Processes and Extensions

In this section, we present a particular construction of random times. This construction is the basic one to define a default time in finance.

2.3.1 Construction of Cox Processes with a given stochastic intensity

Let $(\Omega, \mathcal{G}, \mathbb{P})$ be a probability space endowed with a filtration $\mathbb{F}$. *A nonnegative $\mathbb{F}$-adapted process λ is given.* We assume that there exists, on the space $(\Omega, \mathcal{G}, \mathbb{P})$, a random variable Θ, independent of $\mathcal{F}_\infty$, with an exponential law: $\mathbb{P}(\Theta \geq t) = e^{-t}$. We define the default time τ as the first time when the increasing process $\Lambda_t = \int_0^t \lambda_s \, ds$ is above the random level Θ, i.e.,

$$\tau = \inf \{t \geq 0 \, : \, \Lambda_t \geq \Theta\}.$$

In particular, using the increasing property of Λ, one gets $\{\tau > s\} = \{\Lambda_s < \Theta\}$. We assume that $\Lambda_t < \infty, \forall t, \, \Lambda_\infty = \infty$, hence τ is a real-valued r.v.. One can also define τ as

$$\tau = \inf \{t \geq 0 \, : \, \Lambda_t \geq -\ln U\}$$

where U has a uniform law and is independent of $\mathcal{F}_\infty$. Indeed, the r.v. $-\ln U$ has an exponential law of parameter 1, since $\{-\ln U > a\} = \{U < e^{-a}\}$.

Comment 2.3.1 (i) In order to construct the r.v. Θ, one needs to enlarge the probability space as follows. Let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ be an auxiliary probability space with a r.v. Θ with exponential law. We introduce the product probability space $(\widetilde{\Omega}, \widetilde{\mathcal{G}}, \widetilde{\mathbb{Q}}) = (\Omega \times \hat{\Omega}, \mathcal{F}_\infty \otimes \hat{\mathcal{F}}, \mathbb{Q} \otimes \hat{\mathbb{P}})$.
(ii) Another construction for the default time τ is to choose $\tau = \inf \{t \geq 0 \, : \, \tilde{N}_{\Lambda_t} = 1\}$, where $\Lambda_t = \int_0^t \lambda_s \, ds$ and $\tilde{N}$ is a Poisson process with intensity 1, independent of the filtration $\mathbb{F}$. This second method is in fact equivalent to the first. Cox processes are used in a great number of studies (see, e.g., [89])

2.3.2 Conditional Expectations

Lemma 2.3.2 *The conditional distribution function of τ given the σ-field $\mathcal{F}_t$ is for $t \geq s$*

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \exp\Big(-\Lambda_s\Big).$$

PROOF: The proof follows from the equality $\{\tau > s\} = \{\Lambda_s < \Theta\}$. From the independence assumption and the $\mathcal{F}_t$-measurability of Λ_s for $s \leq t$, we obtain

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \mathbb{P}\Big(\Lambda_s < \Theta \Big| \mathcal{F}_t\Big) = \exp\Big(-\Lambda_s\Big).$$

In particular, we have

$$\mathbb{P}(\tau \leq t|\mathcal{F}_t) = \mathbb{P}(\tau \leq t|\mathcal{F}_\infty), \tag{2.3.1}$$

and, for $t \geq s$, $\mathbb{P}(\tau > s|\mathcal{F}_t) = \mathbb{P}(\tau > s|\mathcal{F}_s)$. Let us notice that the process $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$ is here an increasing process, as the right-hand side of (2.3.1) is. □

Remark 2.3.3 If the process λ is not non-negative, we get,

$$\{\tau > s\} = \{\sup_{u \leq s} \Lambda_u < \Theta\},$$

hence for $s < t$

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \exp(-\sup_{u \leq s} \Lambda_u).$$

More generally, some authors define the default time as

$$\tau = \inf\{t \geq 0 : X_t \geq \Theta\}$$

where X is a given $\mathbb{F}$-semi-martingale. Then, for $s \leq t$

$$\mathbb{P}(\tau > s|\mathcal{F}_t) = \exp(-\sup_{u \leq s} X_u).$$

Exercise 2.3.4 Prove that τ is independent of $\mathcal{F}_\infty$ if and only if λ is a deterministic function. ◁

2.3.3 Choice of filtration

We write as usual $H_t = 1\!\!1_{\{\tau \leq t\}}$ and $\mathcal{H}_t = \sigma(H_s : s \leq t)$. We introduce the smallest right-continuous filtration $\mathbb{G}$ which contains $\mathbb{F}$ and turns τ in a stopping time. (We denote by $\mathbb{F}$ the original **F**iltration and by $\mathbb{G}$ the enlar**G**ed one.) We shall frequently write $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$.

It is easy to describe the events which belong to the σ-field $\mathcal{G}_t$ on the set $\{\tau > t\}$. Indeed, if $G_t \in \mathcal{G}_t$, then $G_t \cap \{\tau > t\} = B_t \cap \{\tau > t\}$ for some event $B_t \in \mathcal{F}_t$.

Therefore any $\mathcal{G}_t$-measurable random variable Y_t satisfies $1\!\!1_{\{\tau > t\}} Y_t = 1\!\!1_{\{\tau > t\}} y_t$, where y_t is an $\mathcal{F}_t$-measurable random variable.

2.3.4 Immersion property

Lemma 2.3.5 *Let X be an $\mathcal{F}_\infty$-measurable integrable r.v.. Then*

$$\mathbb{E}(X|\mathcal{G}_t) = \mathbb{E}(X|\mathcal{F}_t). \tag{2.3.2}$$

PROOF: To prove that $\mathbb{E}(X|\mathcal{G}_t) = \mathbb{E}(X|\mathcal{F}_t)$, it suffices to check that

$$\mathbb{E}(B_t h(\tau \wedge t) X) = \mathbb{E}(B_t h(\tau \wedge t)\mathbb{E}(X|\mathcal{F}_t))$$

for any $B_t \in \mathcal{F}_t$ and any $h = 1\!\!1_{[0,a]}$. For $t \leq a$, the equality is obvious. For $t > a$, we have from (2.3.1)

$$\begin{aligned} \mathbb{E}(B_t 1\!\!1_{\{\tau \leq a\}} \mathbb{E}(X|\mathcal{F}_t)) &= \mathbb{E}(\mathbb{E}(B_t X|\mathcal{F}_t)\mathbb{E}(1\!\!1_{\{\tau \leq a\}}|\mathcal{F}_t)) = \mathbb{E}(X B_t \mathbb{E}(1\!\!1_{\{\tau \leq a\}}|\mathcal{F}_t)) \\ &= \mathbb{E}(B_t X \mathbb{E}(1\!\!1_{\{\tau \leq a\}}|\mathcal{F}_\infty)) = \mathbb{E}(B_t X 1\!\!1_{\{\tau \leq a\}}) \end{aligned}$$

as expected. □

Remark 2.3.6 Let us remark that (2.3.2) implies that every $\mathbb{F}$-square integrable martingale is a $\mathbb{G}$-martingale. However, equality (2.3.2) does not apply to any $\mathcal{G}_\infty$-measurable random variable; in particular $\mathbb{P}(\tau \leq t|\mathcal{G}_t) = 1\!\!1_{\{\tau \leq t\}}$ is not equal to $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$.

This lemma implies that any (u.i.) $\mathbb{F}$ martingale is a $\mathbb{G}$ martingale. This property is known as the immersion property of $\mathbb{F}$ with respect to $\mathbb{G}$ and will be studied in the next chapter. Let us give another proof of this result

Lemma 2.3.7 *In a Cox model, any $\mathbb{F}$ martingale is a $\mathbb{G}$ martingale*

PROOF: Since Θ is independent from $\mathbb{F}$, it is obvious that any $\mathbb{F}$ martingale M is an $\mathbb{F}^\tau = \mathbb{F} \vee \sigma(\Theta)$ martingale. Since $\mathbb{G} \subset \mathbb{F}^\tau$, it follows that M is a $\mathbb{G}$ martingale. □

2.3.5 Extension

One can define the time of default as

$$\tau = \inf\{t \,:\, \Lambda_t \geq \Sigma\}$$

where Σ a non-negative r.v. independent of $\mathcal{F}_\infty$. This model reduces to the previous one: if Φ is the cumulative function of Σ, the r.v. $\Phi(\Sigma)$ has a uniform distribution and

$$\tau = \inf\{t \,:\, \Phi(\Lambda_t) \geq \Phi(\Sigma)\} = \inf\{t \,:\, \Psi^{-1}[\Phi(\Lambda_t)] \geq \Theta\}$$

where Ψ is the cumulative function of the exponential law. Then,

$$F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t) = \mathbb{P}(\Lambda_t \geq \Sigma|\mathcal{F}_t) = 1 - \exp\left(-\Psi^{-1}(\Phi(\Lambda_t))\right) .$$

2.3.6 Dynamics of prices in a default setting

We assume here that $\mathbb{F}$-martingales are continuous.

Defaultable Zero-Coupon Bond

A defaultable Zero-coupon Bond of maturity T pays one monetary unit at time T, if the default has not occurred before T. Let $\mathbb{Q}$ be a risk-neutral probability and $B(t,T)$ be the price at time t of a default-free bond paying 1 at maturity T satisfies

$$B(t,T) = \mathbb{E}_{\mathbb{Q}}\Big(\exp\big(- \int_t^T r_s\, ds\big) \,\Big|\, \mathcal{F}_t \Big).$$

The market price $D(t,T)$ of a defaultable zero-coupon bond with maturity T is

$$\begin{aligned} D(t,T) &= \mathbb{E}_{\mathbb{Q}}\Big(1\!\!1_{\{T<\tau\}} \exp\big(-\int_t^T r_s\, ds\big) \,\Big|\, \mathcal{G}_t\Big) \\ &= 1\!\!1_{\{\tau>t\}} \mathbb{E}_{\mathbb{Q}}\Big(\exp\big(-\int_t^T [r_s+\lambda_s]\, ds\big) \,\Big|\, \mathcal{F}_t\Big). \end{aligned}$$

Here, the process λ is defined under $\mathbb{Q}$ (i.e. $\mathbb{Q}(\tau > t|\mathcal{F}_t) = \exp - \int_0^t \lambda_s ds$). Then, in the case $r = 0$,

$$D(t,T) = 1\!\!1_{t<\tau} e^{\Lambda_t} \mathbb{Q}(\tau > T|\mathcal{F}_t) = L_t m_t$$

with $m_t = \mathbb{Q}(\tau > T|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}(e^{-\Lambda_T}|\mathcal{F}_t)$. Then,

$$dD(t,T) = m_t dL_t + L_{t-} dm_t = -m_t L_{t-} dM_t + L_{t-} dm_t = -D(t-,T) dM_t + L_{t-} dm_t$$

In the particular case where λ is deterministic, $m_t = e^{-\Lambda_T}$ and $dm_t = 0$. Hence $D(t,T) = L_t e^{-\Lambda_T}$ and

$$dD(t,T) = -D(t-,T) dM_t \,.$$

Remark 2.3.8 If $\mathbb{P}$ is a probability such that Θ is independent of $\mathcal{F}_\infty$ and $\mathbb{Q}$ a probability equivalent to $\mathbb{P}$, it is not true, in general that Θ is independent of $\mathcal{F}_\infty$ and has an exponential law under $\mathbb{Q}$. Changes of probabilities that preserve the independence of Θ and $\mathcal{F}_\infty$ change the law of Θ, hence the intensity.

Exercise 2.3.9 Write the risk-neutral dynamics of D for a general interest rate r. $\triangleleft$

Recovery with Payment at maturity

We assume here that $r = 0$. We consider a contract which pays K_τ at date T, if $\tau \le T$ and no payment in the case $\tau > T$, where K is a given $\mathbb{F}$-predictable process.

The price at time t of this contract is

$$\begin{aligned} S_t &= E(K_\tau 1\!\!1_{\tau<T}|\mathcal{G}_t) = K_\tau 1\!\!1_{\tau<t} + 1\!\!1_{t<\tau} E(K_\tau 1\!\!1_{t<\tau<T}|\mathcal{G}_t) \\ &= K_\tau 1\!\!1_{\tau<t} + 1\!\!1_{t<\tau} e^{\Lambda_t} E(\int_t^T K_u dF_u|\mathcal{F}_t) \end{aligned}$$

where $F_u = P(\tau \le u|\mathcal{F}_u) = 1 - e^{-\Lambda_u}$, or

$$S_t = K_\tau 1\!\!1_{\tau<t} + 1\!\!1_{t<\tau} e^{\Lambda_t} E(\int_t^T K_u e^{-\Lambda_u} \lambda_u du|\mathcal{F}_t)$$

or

$$S_t = \int_0^t K_u dH_u + L_t \left(- \int_0^t K_u e^{-\Lambda_u} \lambda_u du + m_t^K \right)$$

where $m_t^K = E(\int_0^T K_u e^{-\Lambda_u} \lambda_u du|\mathcal{F}_t)$. From $dL_t = -L_{t-} dM_t$ and

$$d(Lm^K)_t = L_{t-} dm_t^K + m_{t-}^K dL_t + d[m^K, L]_t = L_{t-} dm_t^K + m_{t-}^K dL_t$$

we deduce that

$$dS_t = K_t (dH_t - \lambda_t (1 - H_t) dt) - S_{t-} dM_t + L_t dm_t^K = (K_t - S_{t-}) dM_t + L_t dm_t^K$$

Note that, since m^K is continuous, its covariation process with L is null and that one can write $L_t dm_t^K$ instead of $L_{t-} dm_t^K$. Note also that, from the definition, the process S is a $\mathbb{G}$-martingale. This can be checked looking at the dynamics, since m^K is a $\mathbb{F}$, hence a $\mathbb{G}$, martingale (WHY?)

Exercise 2.3.10 Write the risk-neutral dynamics of the price of the recovery for a general interest rate r. $\triangleleft$

Recovery with Payment at Default time

Let K be a given $\mathbb{F}$-predictable process. The payment K_τ is done at time τ. Then, in the case $r = 0$,

$$S_t = 1\!\!1_{t<\tau} E(K_\tau 1\!\!1_{t<\tau<T}|\mathcal{G}_t) = 1\!\!1_{t<\tau} e^{\Lambda_t} E(\int_t^T K_u dF_u|\mathcal{F}_t)\,.$$

The dynamics of S is

$$dS_t = -S_{t-}dM_t + L_t(dm_t^K - K_t e^{-\Lambda_t}\lambda_t)dt = -S_{t-}dM_t + (1-H_t)(e^{\Lambda_t}dm_t^K - K_t\lambda_t)dt$$

and the process $S_t + K_\tau 1\!\!1_{\{\tau<t\}} = S_t + \int_0^t K_s dH_s = E(K_\tau|\mathcal{G}_t)$ is a $\mathbb{G}$-martingale, as well as the process $S_t + \int_0^{t\wedge\tau} K_s\lambda_s ds$. The quantity $K_t\lambda_t$ which appears in the dynamics of S can be interpreted as a dividend K_t paid at rate λ_t (or with probability $\lambda_t dt = P(t<\tau<t+dt|\mathcal{F}_t)/P(t<\tau|\mathcal{F}_t)$)

Price and Hedging a Defaultable Call

We assume that

- the savings account $Y_t^0 = 1$
- a risky asset with risk-neutral dynamics

  $$dY_t = Y_t\sigma dW_t$$

  where W is a Brownian motion and σ is a constant
- a DZC of maturity T with price $D(t,T)$

are traded. The reference filtration is that of the BM W. The price of a defaultable call with payoff $1\!\!1_{T<\tau}(Y_T-K)^+$ is

$$\begin{aligned} C_t &= \mathbb{E}(1\!\!1_{T<\tau}(Y_T-K)^+|\mathcal{G}_t) = 1\!\!1_{t<\tau}e^{\Lambda_t}\mathbb{E}(e^{-\Lambda_T}(Y_T-K)^+|\mathcal{F}_t) \\ &= L_t m_t^Y \end{aligned}$$

with $m_t^Y = \mathbb{E}(e^{-\Lambda_T}(Y_T-K)^+|\mathcal{F}_t)$. Hence

$$dC_t = L_t dm_t^Y - m_t^Y L_{t-}dM_t$$

- In the particular case where λ is deterministic,

$$m_t^Y = e^{-\Lambda_T}E((Y_T-K)^+|\mathcal{F}_t) = e^{-\Lambda_T}C_t^Y$$

where C^Y is the price of a call in the Black Scholes model. This quantity is $C_t^Y = C^Y(t,Y_t)$ and satisfies $dC_t^Y = \Delta_t dY_t$ where $\Delta_t$is the Delta-hedge ($\Delta_t = \partial_y C^Y(t,Y_t)$)

$$C_t = 1\!\!1_{t<\tau}e^{\Lambda_t}e^{-\Lambda_T}C^Y(t,Y_t) = L_t e^{-\Lambda_T}C^Y(t,Y_t) = D(t,T)C^Y(t,Y_t)$$

From

$$C_t = D(t,T)C^Y(t,Y_t)$$

we deduce

$$\begin{aligned} dC_t &= e^{-\Lambda_T}(L_t dC^Y + C^Y dL_t) = e^{-\Lambda_T}(L_t\Delta_t dY_t - C^Y L_t dM_t) \\ &= e^{-\Lambda_T}(L_t\Delta_t dY_t - C^Y L_t dM_t) \end{aligned}$$

Therefore, using that $dD(t,T) = m_t dM_t = -e^{-\Lambda_T}L_t dM_t$ we get

$$dC_t = e^{-\Lambda_T}L_t\Delta_t dY_t - C^Y dD(t,T) = e^{-\Lambda_T}L_t\Delta_t dY_t + \frac{C_t}{D(t,T)}dD(t,T)$$

hence, an hedging strategy consists of holding in particular $\frac{C_t}{D(t,T)}$ DZCs.

• In the general case, one obtains

$$dC_t = \frac{C_{t-}}{D(t,T)} dD(t,T) + L_t \frac{m_t^Y}{m_t} dm_t + L_t dm_t^Y = \frac{C_{t-}}{D(t,T)} dD(t,T) + \vartheta_t dY_t$$

An hedging strategy consists of holding $\frac{C_{t-}}{D(t,T)}$ DZCs.

Credit Default Swap

Definition 2.3.11 *A T-maturity credit default swap (CDS) with a constant rate κ and recovery at default is a contract. The seller agrees to pay the recovery at default time, the buyer pays (in continuous time) the premium κ till maturity or to default time, whichever occurs the first. The $\mathbb{F}$-predictable process $\delta : [0,T] \to \mathbb{R}$ represents the default protection, and the constant κ is the fixed CDS rate (also termed the spread or premium of the CDS).*

Let $B_t = \exp \int_0^t r_s ds$. The cumulative ex-dividend price of a CDS equals, for any $t \in [0,T]$, to the expectation of the remaining discounted future payoffs

$$S_t = B_t \mathbb{E}_{\mathbb{Q}}((B_\tau)^{-1} \delta_\tau 1\!\!1_{t<\tau \le T} - \int_t^{T\wedge\tau} \kappa B_s^{-1} ds | \mathcal{G}_t)$$

The cumulative price is

$$S_t = B_t \mathbb{E}_{\mathbb{Q}}((B_\tau)^{-1} \delta_\tau 1\!\!1_{\tau \le T} - \int_0^{T\wedge\tau} \kappa B_s^{-1} ds | \mathcal{G}_t)$$

We denote by D the dividend process associated with the CDS:

$$D_t = Z_t 1\!\!1_{\tau \le t} - \kappa(t \wedge \tau)$$

An immediate application of the key lemma gives the following result

Proposition 2.3.12 *The ex-dividend price of a CDS equals, for any $t \in [0,T]$,*

$$S_t(\kappa) = 1\!\!1_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}} \left(\int_t^T B_u^{-1} G_u \delta_u \lambda_u \, du - \kappa \int_t^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_t \right), \tag{2.3.3}$$

and thus the cumulative price of a CDS equals, for any $t \in [0,T]$,

$$S_t^{\mathrm{cum}}(\kappa) = 1\!\!1_{\{t<\tau\}} \frac{B_t}{G_t} \mathbb{E}_{\mathbb{Q}} \left(\int_t^T B_u^{-1} G_u \delta_u \lambda_u \, du - \kappa \int_t^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_t \right) + B_t \int_{]0,t]} B_u^{-1} \, dD_u. \tag{2.3.4}$$

An easy computation yields to

Corollary 2.3.13 *The dynamics of the ex-dividend price $S(\kappa)$ on $[0,T]$ are*

$$dS_t(\kappa) = -S_{t-}(\kappa) \, dM_t + (1-H_t)\big(r_t S_t + \kappa - \lambda_t \delta_t\big) \, dt + (1-H_t) G_t^{-1} B_t \, dn_t,$$

where the $\mathbb{F}$-martingale n is given by the formula

$$n_t = \mathbb{E}_{\mathbb{Q}} \left(\int_0^T B_u^{-1} G_u \delta_u \lambda_u \, du - \kappa \int_0^T B_u^{-1} G_u \, du \,\Big|\, \mathcal{F}_t \right). \tag{2.3.5}$$

The dynamics of the cumulative price $S^{\mathrm{cum}}(\kappa)$ on $[0,T]$ are

$$dS_t^{\mathrm{cum}}(\kappa) = r_t S_t^{\mathrm{cum}}(\kappa) \, dt + \big(\delta_t - S_{t-}(\kappa)\big) \, dM_t + (1-H_t) G_t^{-1} B_t \, dn_t$$

2.3.7 Generalisation

We start with the filtered space $(\Omega, \mathcal{G}, \mathbb{F}, \mathbb{P})$ and the random variable Θ, independent of $\mathcal{F}_\infty$, with an exponential law. We define the default time τ as the first time when the increasing process Γ is above the random level Θ, i.e.,

$$\tau = \inf\{t \geq 0 \,:\, \Gamma_t \geq \Theta\}.$$

We do not assume any more that Γ is absolutely continuous, and we are even interested with the case where Γ fails to be continuous.

The same proof as before yields to

$$\mathbb{P}(\tau > t|\mathcal{F}_t) = e^{-\Gamma_t}$$

However, since Γ fails to be predictable, the compensator of H is no more Γ

Let us study the following example. Let X be a compound Poisson process, with positive jumps, i.e.,

$$X_t = \sum_{n=1}^{N_t} Y_n$$

where N is a Poisson process and Y_n positive random variable, i.i.d. and independent from N.

Let $\psi(u) = \int_0^\infty (1 - e^{uy}) F(dy)$ where F is the cumulative distribution function of Y_1. Then, $e^{uX_t + t\lambda\psi(u)}$ is a martingale. Then, from $G_t = e^{-X_t} = e^{-X_t + t\lambda\psi(-1)} e^{-t\lambda\psi(-1)} = n_t e^{-t\lambda\psi(-1)}$ where n is a martingale one deduce, by integration by parts the Doob-Meyer decomposition that

$$dG_t = e^{-t\lambda\psi(-1)} dN_t - e^{-t\lambda\psi(-1)} n_t \lambda\psi(-1) dt$$

and it follows that

$$\mathbb{1}_{\tau \leq t} - (t \wedge \tau)\lambda\psi(-1)$$

is a martingale.

One can also compute directly the Doob-Meyer decomposition of supermartingale G from Itô's formula. Let μ the jump measure of X

$$e^{-X_t} = 1 + \int_0^t \int (e^{-(X_u + y)} - e^{-X_u}) \mu(du, ds) - du\lambda F(dy) + \int_0^t \int (e^{-(X_u+y)} - e^{-X_u}) du\lambda F(dy)$$

where the quantity $\int_0^t \int (e^{-(X_u+y)} - e^{-X_u})(\mu(du, ds) - du\lambda F(dy))$ represents a martingale. Hence the form of the compensator.

Chapter 3

Generalities and Immersion Property

From the end of the seventies, Jacod, Jeulin and Yor started a systematic study of the problem of enlargement of filtrations: namely, if $\mathbb{F}$ and $\mathbb{G}$ are two filtrations satisfying $\mathbb{F} \subset \mathbb{G}$, which $\mathbb{F}$-martingales M remain $\mathbb{G}$-semi-martingales and if it is the case, what is the semi-martingale decomposition of M in $\mathbb{G}$?

In the literature, there are mainly two kinds of enlargement of filtration:

• Initial enlargement of filtrations: in that case, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L)$ where L is a r.v. (or, more generally $\mathcal{G}_t = \mathcal{F}_t \vee \widetilde{\mathcal{F}}$ where $\widetilde{\mathcal{F}}$ is a σ-algebra, up to right-continuous regularization)

• Progressive enlargement of filtrations, where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ with $\mathbb{H}$ the natural filtration of $H_t = 1\!\!1_{\{\tau \leq t\}}$ where τ is a random time (or, more generally $\mathcal{G}_t = \mathcal{F}_t \vee \widetilde{\mathcal{F}}_t$ where $\widetilde{\mathbb{F}}$ is another filtration). In fact, very few studies are done in the case $\mathcal{G}_t = \mathcal{F}_t \vee \widetilde{\mathcal{F}}_t$. One exception is for $\widetilde{\mathcal{F}}_t = \sigma(J_t)$ where $J_t = \inf_{s \geq t} X_s$ when X is a three dimensional Bessel process (see [73]). See also the recent work of Kchia et al. [80].

Up to now, three lecture notes volumes have been dedicated to this question: Jeulin [73], Jeulin & Yor [77] and Mansuy & Yor [93]. There are also related chapters in the books of Protter [101] Dellacherie, Maisonneuve & Meyer [35], Jacod [64], Jeanblanc et al. [72] and Yor [115].

Some first and important papers are Brémaud and Yor [25] (devoted to immersion case), Barlow [14] (for a specific study of honest times), Jacod [64, 65] and Jeulin & Yor [74]. A non-exhaustive list of references contains the papers of Ankirchner et al. [9], Nikeghbali [97] and Yoeurp [112].

Several thesis are devoted to this problem: Aksamit [1] Amendinger [5], Ankirchner [8], Bedini [17], Kchia [81], Kreher [86], Li [90], Song [105] and Wu [?].

Enlargement of filtration results are extensively used in finance to study two specific problems occurring in insider trading: existence of arbitrage using strategies adapted w.r.t. the large filtration, and change of prices dynamics, when an $\mathbb{F}$-martingale is no longer a $\mathbb{G}$-martingale. They are also a main stone for study of default risk.

An incomplete list of authors concerned with enlargement of filtration in finance for insider trading is: Ankirchner [9, 8], Amendinger [5, 6], Amendinger et al. [7], Baudoin [16], Corcuera et al. [31], Eyraud-Loisel [47], Florens & Fougère [51], Gasbarra et al. [56], Grorud & Pontier [57], Hillairet [60], Imkeller [61], Karatzas & Pikovsky [79], Wu [111], Kohatsu-Higa & Øksendal [85], Zwierb [].

A general study of arbitrages which can occur in an enlarged filtration is presented in Aksamit et al. [3, 4, 2, ?], Acciao et al. [12], Fontana et al. [55]

Di Nunno et al. [40], Imkeller [62], Imkeller et al. [63], Kohatsu-Higa [83, 84] have introduced Malliavin calculus to study the insider trading problem. We shall not discuss this approach here.

Enlargement theory is also used to study asymmetric information, see, e.g. Föllmer et al. [54] and progressive enlargement is an important tool for the study of default in the reduced form approach by Bielecki et al. [20, 21, 22], Elliott et al.[45] and Kusuoka [88] among others.

Let $\mathbb{F}$ and $\mathbb{G}$ be two filtrations such that $\mathbb{F} \subset \mathbb{G}$. Our aim is to study some conditions which ensure that $\mathbb{F}$-martingales are $\mathbb{G}$-semi-martingales, and one can ask in a first step whether all $\mathbb{F}$-martingales are $\mathbb{G}$-martingales. This last property is equivalent to $\mathbb{E}(\zeta|\mathcal{F}_t) = \mathbb{E}(\zeta|\mathcal{G}_t)$, for any t and $\zeta \in L^1(\mathcal{F}_\infty)$.

Let us study a simple example where $\mathbb{G} = \mathbb{F} \vee \sigma(\zeta)$ where $\zeta \in L^1(\mathcal{F}_\infty)$ and ζ is not $\mathcal{F}_0$-measurable. Obviously, $m_t := \mathbb{E}(\zeta|\mathcal{F}_t)$ is an $\mathbb{F}$-martingale. If m would be a $\mathbb{G}$-martingale, $\mathbb{E}(m_\infty|\mathcal{G}_t) = m_t$, hence $\zeta = m_t$ and, in particular $\zeta = \mathbb{E}(\zeta|\mathcal{F}_0)$ which is not the case.

In this chapter, we start with the case where $\mathbb{F}$-martingales remain $\mathbb{G}$-martingales. In that case, there is a complete characterization so that this property holds. Then, we study a particular example: Brownian and Poisson bridges.

3.1 Immersion of Filtrations

3.1.1 Definition

The filtration $\mathbb{F}$ is said to be **immersed** in $\mathbb{G}$ if any square integrable $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale (Tsirel'son [109], Émery [46]). This is also referred to as the $(\mathcal{H})$ hypothesis by Brémaud and Yor [25].

$(\mathcal{H})$ Every $\mathbb{F}$-square integrable martingale is a $\mathbb{G}$-square integrable martingale.

Proposition 3.1.1 *Hypothesis $(\mathcal{H})$ is equivalent to any of the following properties:*

- $(\mathcal{H}1)$ $\forall\, t \geq 0$, *the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$, i.e.,* $\forall\, t \geq 0,\ \forall\, G_t \in L^2(\mathcal{G}_t), \forall\, F \in L^2(\mathcal{F}_\infty), \mathbb{E}(G_t F|\mathcal{F}_t) = \mathbb{E}(G_t|\mathcal{F}_t)\mathbb{E}(F|\mathcal{F}_t)$.
- $(\mathcal{H}2)$ $\forall\, t \geq 0,\ \forall\, G_t \in L^1(\mathcal{G}_t),\ \mathbb{E}(G_t|\mathcal{F}_\infty) = \mathbb{E}(G_t|\mathcal{F}_t)$.
- $(\mathcal{H}3)$ $\forall\, t \geq 0,\ \forall\, F \in L^1(\mathcal{F}_\infty),\ \mathbb{E}(F|\mathcal{G}_t) = \mathbb{E}(F|\mathcal{F}_t)$.

In particular, $(\mathcal{H})$ holds if and only if every $\mathbb{F}$-local martingale is a $\mathbb{G}$-local martingale.

PROOF:
• $(\mathcal{H}) \Rightarrow (\mathcal{H}1)$. Let $F \in L^2(\mathcal{F}_\infty)$ and assume that hypothesis $(\mathcal{H})$ is satisfied. This implies that the martingale $F_t = \mathbb{E}(F|\mathcal{F}_t)$ is a $\mathbb{G}$-martingale such that $F_\infty = F$, hence $F_t = \mathbb{E}(F|\mathcal{G}_t)$. It follows that for any t and any $G_t \in L^2(\mathcal{G}_t)$:

$$\mathbb{E}(FG_t|\mathcal{F}_t) = \mathbb{E}(G_t\mathbb{E}(F|\mathcal{G}_t)|\mathcal{F}_t) = \mathbb{E}(G_t\mathbb{E}(F|\mathcal{F}_t)|\mathcal{F}_t) = \mathbb{E}(G_t|\mathcal{F}_t)\mathbb{E}(F|\mathcal{F}_t)$$

which is exactly $(\mathcal{H}1)$.
• $(\mathcal{H}1) \Rightarrow (\mathcal{H}2)$. Let $F \in L^2(\mathcal{F}_\infty)$ and $G_t \in L^2(\mathcal{G}_t)$. Under $(\mathcal{H}1)$,

$$\mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_t)) = \mathbb{E}(\mathbb{E}(F|\mathcal{F}_t)\mathbb{E}(G_t|\mathcal{F}_t)) \overset{\mathcal{H}1}{=} \mathbb{E}(\mathbb{E}(FG_t|\mathcal{F}_t)) = \mathbb{E}(FG_t)$$

which is $(\mathcal{H}2)$.
• $(\mathcal{H}2) \Rightarrow (\mathcal{H}3)$. Let $F \in L^2(\mathcal{F}_\infty)$ and $G_t \in L^2(\mathcal{G}_t)$. If $(\mathcal{H}2)$ holds, then it is easy to prove that, for $F \in L^2(\mathcal{F}_\infty)$,

$$\mathbb{E}(G_t\mathbb{E}(F|\mathcal{F}_t)) = \mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_t)) \overset{\mathcal{H}2}{=} \mathbb{E}(F\mathbb{E}(G_t|\mathcal{F}_\infty))\mathbb{E}(FG_t),$$

which implies $(\mathcal{H}3)$.
• Obviously $(\mathcal{H}3)$ implies $(\mathcal{H})$. □

In particular,if $\mathbb{F}$ is immersed in $\mathbb{G}$ and if W is an $\mathbb{F}$-Brownian motion, then it is a $\mathbb{G}$-martingale with bracket t, since such a bracket does not depend on the filtration. Hence, it is a $\mathbb{G}$-Brownian motion. It is important to note that $\int_0^t \psi_s dW_s$ is then a $\mathbb{G}$-local martingale, for a $\mathbb{G}$-adapted process ψ, satisfying some integrability conditions (see [76]).

A trivial (but useful) example for which $\mathbb{F}$ is immersed in $\mathbb{G}$ is $\mathbb{G} = \mathbb{F} \vee \widetilde{\mathbb{F}}$ where $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ are two filtrations such that $\mathcal{F}_\infty$ is independent of $\widetilde{\mathcal{F}}_\infty$.

Exercise 3.1.2 Assume that $\mathbb{F}$ is immersed in $\mathbb{G}$ and that W is an $\mathbb{F}$-Brownian motion. Prove that W is a $\mathbb{G}$-Brownian motion without using the bracket. ◁

Exercise 3.1.3 Prove that, if $\mathbb{F}$ is immersed in $\mathbb{G}$, then, for any t, $\mathcal{F}_t = \mathcal{G}_t \cap \mathcal{F}_\infty$. ◁

Exercise 3.1.4 Show that, if $\tau \in \mathcal{F}_\infty$, immersion holds between $\mathbb{F}$ and $\mathbb{F} \vee \mathbb{H}$ where $\mathbb{H}$ is generated by $H_t = 1\!\!1_{\tau \le t}$ if and only if τ is an $\mathbb{F}$-stopping time. ◁

3.1.2 Change of probability

Of course, the notion of immersion depends strongly on the probability measure, and in particular, is not stable by change of probability. See Subsection 3.3.5 for a counter example. We now study in which setup the immersion property is preserved under change of probability.

Proposition 3.1.5 *Assume that the filtration $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$, and let $\mathbb{Q}$ be equivalent to $\mathbb{P}$, with $\mathbb{Q}|_{\mathcal{G}_t} = L_t \mathbb{P}|_{\mathcal{G}_t}$ where L is assumed to be $\mathbb{F}$-adapted. Then, $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{Q}$.*

Proof: Let N be a $(\mathbb{F},\mathbb{Q})$-martingale, then $(N_t L_t, t \ge 0)$ is a $(\mathbb{F},\mathbb{P})$-martingale, and since $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$, $(N_t L_t, t \ge 0)$ is a $(\mathbb{G},\mathbb{P})$-martingale which implies that N is a $(\mathbb{G},\mathbb{Q})$-martingale. □

Note that, if one defines a change of probability on $\mathbb{F}$ with a Radon-Nikodým density which is (as it must be) an $\mathbb{F}$-martingale L, one can not extend this change of probability to $\mathbb{G}$ by setting $\mathbb{Q}|_{\mathcal{G}_t} = L_t \mathbb{P}|_{\mathcal{G}_t}$, since, in general, L fails to be a $\mathbb{G}$-martingale.

In the next proposition, we do not assume that the Radon-Nikodým density is $\mathbb{F}$-adapted.

Proposition 3.1.6 *Assume that $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$, and let $\mathbb{Q}$ be equivalent to $\mathbb{P}$ with $\mathbb{Q}|_{\mathcal{G}_t} = L_t \mathbb{P}|_{\mathcal{G}_t}$ where L is a $\mathbb{G}$-martingale and define $\ell_t = \mathbb{E}(L_t|\mathcal{F}_t)$. Assume that all $\mathbb{F}$-martingales are continuous and that L is continuous. Then, $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{Q}$ if and only if the $(\mathbb{G},\mathbb{P})$-local martingale*

$$\int_0^t \frac{dL_s}{L_s} - \int_0^t \frac{d\ell_s}{\ell_s} := \mathcal{L}(L)_t - \mathcal{L}(\ell)_t$$

is orthogonal to the set of all $(\mathbb{F},\mathbb{P})$-local martingales.

Proof: Every $(\mathbb{F},\mathbb{Q})$-martingale $M^{\mathbb{Q}}$ may be written as

$$M_t^{\mathbb{Q}} = M_t^{\mathbb{P}} - \int_0^t \frac{d\langle M^{\mathbb{P}}, \ell\rangle_s}{\ell_s}$$

where $M^{\mathbb{P}}$ is an $(\mathbb{F},\mathbb{P})$-martingale. By immersion hypothesis, $M^{\mathbb{P}}$ is a $(\mathbb{G},\mathbb{P})$-martingale and, from Girsanov's theorem, $M_t^{\mathbb{P}} = N_t^{\mathbb{Q}} + \int_0^t \frac{d\langle M^{\mathbb{P}}, L\rangle_s}{L_s}$ where $N^{\mathbb{Q}}$ is an $(\mathbb{G},\mathbb{Q})$-martingale. It follows that

$$\begin{aligned} M_t^{\mathbb{Q}} &= N_t^{\mathbb{Q}} + \int_0^t \frac{d\langle M^{\mathbb{P}}, L\rangle_s}{L_s} - \int_0^t \frac{d\langle M^{\mathbb{P}}, \ell\rangle_s}{\ell_s} \\ &= N_t^{\mathbb{Q}} + \int_0^t d\langle M^{\mathbb{P}}, \mathcal{L}(L) - \mathcal{L}(\ell)\rangle_s . \end{aligned}$$

Thus $M^{\mathbb{Q}}$ is a $(\mathbb{G},\mathbb{Q})$ martingale if and only if $\langle M^{\mathbb{P}}, \mathcal{L}(L)-\mathcal{L}(\ell)\rangle_s = 0$. Here $\mathcal{L}$ is the stochastic logarithm, as the inverse of the stochastic exponential. □

Proposition 3.1.7 *Let $\mathbb{P}$ be a probability measure, and*

$$\mathbb{Q}|_{\mathcal{G}_t} = L_t \mathbb{P}|_{\mathcal{G}_t}; \quad \mathbb{Q}|_{\mathcal{F}_t} = \ell_t \mathbb{P}|_{\mathcal{F}_t} .$$

Then, hypothesis $(\mathcal{H})$ holds under $\mathbb{Q}$ if and only if:

$$\forall T, \forall X \geq 0,\ X \in \mathcal{F}_T, \forall t < T, \quad \frac{\mathbb{E}_{\mathbb{P}}(XL_T|\mathcal{G}_t)}{L_t} = \frac{\mathbb{E}_{\mathbb{P}}(X\ell_T|\mathcal{F}_t)}{\ell_t} \tag{3.1.1}$$

PROOF: Note that, for $X \in \mathcal{F}_T$,

$$\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t) \quad = \quad \frac{1}{L_t}\mathbb{E}_{\mathbb{P}}(XL_T|\mathcal{G}_t) \quad , \quad \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t) = \frac{1}{\ell_t}\mathbb{E}_{\mathbb{P}}(X\ell_T|\mathcal{G}_t)$$

and that, from MCT, $(\mathcal{H})$ holds under $\mathbb{Q}$ if and only if, $\forall T, \forall X \in F_T, \forall t \leq T$, one has

$$\mathbb{E}_{\mathbb{Q}}(X|\mathcal{G}_t) = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t) .$$

□

Comment 3.1.8 The $(\mathcal{H})$ hypothesis was studied by Brémaud and Yor [25] and Mazziotto and Szpirglas [95], and in a financial setting by Kusuoka [88], Elliott et al. [45] and Jeanblanc and Rutkowski [68, 69].

Exercise 3.1.9 Prove that, if $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{P}$ and if $\mathbb{Q}$ is a probability equivalent to $\mathbb{P}$, then, any $(\mathbb{Q},\mathbb{F})$-semi-martingale is a $(\mathbb{Q},\mathbb{G})$-semi-martingale. Let

$$\mathbb{Q}|_{\mathcal{G}_t} = L_t \mathbb{P}|_{\mathcal{G}_t}; \quad \mathbb{Q}|_{\mathcal{F}_t} = \ell_t \mathbb{P}|_{\mathcal{F}_t} .$$

and X be a $(\mathbb{Q},\mathbb{F})$ martingale. Assuming that $\mathbb{F}$ is a Brownian filtration and that L is continuous, prove that

$$X_t + \int_0^t \left(\frac{1}{\ell_s} d\langle X, \ell\rangle_s - \frac{1}{L_s} d\langle X, L\rangle_s\right)$$

is a $(\mathbb{G},\mathbb{Q})$ martingale.
In a general case, prove that

$$X_t + \int_0^t \frac{L_{s-}}{L_s}\left(\frac{1}{\ell_{s^-}} d[X, \ell]_s - \frac{1}{L_{s^-}} d[X, L]_s\right)$$

is a $(\mathbb{G},\mathbb{Q})$ martingale. See Jeulin and Yor [75]. ◁

Exercise 3.1.10 Let $\mathbb{F}$ be immersed in $\mathbb{G}$ and x a $\mathbb{G}$ predictable process, m an $\mathbb{F}$-martingale. Prove that $\mathbb{E}(\int_0^t x_s dm_s|\mathcal{F}_t) = \int_0^t \mathbb{E}(x_s|\mathcal{F}_s) dm_s)$. See Brémaud and Yor [25]. ◁

✓EXERCISE: APPLICATION TO E-TIMES

Exercise 3.1.11 Assume that any $\mathbb{F}$ martingale is a $\widetilde{\mathbb{F}}$ semi-martingale, with $\mathbb{F} \subset \widetilde{\mathbb{F}}$, and τ an $\widetilde{\mathbb{F}}$ stopping time. Prove that any $\mathbb{F}$ martingale is a $\mathbb{G}$ semi-martingale, where $\mathbb{G}_t = \sigma(t \wedge \tau)$ (regulariser) ◁

Exercise 3.1.12 Assume that $\mathbb{F}$ is immersed in $\widetilde{\mathbb{F}}$ and τ is an $\widetilde{\mathbb{F}}$ stopping time. Prove that any $\mathbb{F}$ is immersed in $\mathbb{G}$ (regulariser) ◁

Exercise 3.1.13 Assume that $\mathcal{F}_t^{(L)} = \mathcal{F}_t \vee \sigma(L)$ where L is a random variable. Find under which conditions on L, immersion property holds. $\lhd$

Exercise 3.1.14 Construct an example where some $\mathbb{F}$-martingales are $\mathbb{G}$-martingales, but not all $\mathbb{F}$ martingales are $\mathbb{G}$-martingales. $\lhd$

Exercise 3.1.15 Assume that $\mathbb{F} \subset \widetilde{\mathbb{G}}$ where $(\mathcal{H})$ holds for $\mathbb{F}$ and $\widetilde{\mathbb{G}}$.
a) Let τ be a $\widetilde{\mathbb{G}}$-stopping time. Prove that $(\mathcal{H})$ holds for $\mathbb{F}$ and $\mathbb{F}^\tau = \mathbb{F} \vee \mathbb{H}$ where $\mathcal{H}_t = \sigma(\tau \wedge t)$.
b) Let $\mathbb{G}$ be such that $\mathbb{F} \subset \mathbb{G} \subset \widetilde{\mathbb{G}}$. Prove that $\mathbb{F}$ be immersed in $\mathbb{G}$. $\lhd$

Exercise 3.1.16 Assume that $\mathcal{F}_t^{(\tau)} = \mathcal{F}_t \vee \sigma(\tau)$ where τ is a positive random variable, and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ where $\mathcal{H}_t = \sigma(\tau \wedge t)$. Find under which conditions on τ the filtration $\mathbb{G}$ is immersed in $\mathbb{F}^{(\tau)}$. $\lhd$

3.2 Immersion in a Progressive Enlargement of Filtration

We now consider the case where a random time τ is given and where $\mathbb{G}$ is the progressively enlarged filtration. We introduce the $\mathbb{F}$-supermartingale $Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$.

3.2.1 Characterization of Immersion

Lemma 3.2.1 *In the progressive enlargement setting, $(\mathcal{H})$ holds between $\mathbb{F}$ and $\mathbb{G}$ if and only if one of the following equivalent conditions holds:*

$$\begin{array}{llrcl} \text{(i)} & \forall (t,s),\ s \le t, & \mathbb{P}(\tau \le s|\mathcal{F}_\infty) & = & \mathbb{P}(\tau \le s|\mathcal{F}_t), \\ \text{(ii)} & \forall t, & \mathbb{P}(\tau \le t|\mathcal{F}_\infty) & = & \mathbb{P}(\tau \le t|\mathcal{F}_t). \end{array} \tag{3.2.1}$$

PROOF: If (ii) holds, then (i) holds too. If (i) holds, $\mathcal{F}_\infty$ and $\sigma(t \wedge \tau)$ are conditionally independent given $\mathcal{F}_t$. The property follows. This result can also be found in Dellacherie and Meyer [37]. $\square$

Note that, if $(\mathcal{H})$ holds, then (ii) implies that the process $\mathbb{P}(\tau \le t|\mathcal{F}_t)$ is increasing (See Section 7.5 for a study of that property).

Exercise 3.2.2 Prove that in a Cox model (see Section 2.3), immersion holds. $\lhd$

Exercise 3.2.3 Prove that if $\mathbb{H}$ and $\mathbb{F}$ are immersed in $\mathbb{G}$, and if any $\mathbb{F}$ martingale is continuous, then τ and $\mathcal{F}_\infty$ are independent. $\lhd$

Exercise 3.2.4 Assume that immersion property holds and let, for every u, $y_t(u)$ be an $\mathbb{F}$-martingale. Prove that, for $t > s$,

$$1\!\!1_{\tau \le s}\mathbb{E}(y_t(\tau)|\mathcal{G}_s) = 1\!\!1_{\tau \le s} y_s(\tau)$$

$\lhd$

Exercise 3.2.5 Prove that $\mathbb{G}$ is immersed in $\mathbb{F} \vee \sigma(\tau)$ if and only if τ is constant. $\lhd$

3.2.2 Norros's lemma

Proposition 3.2.6 *Assume that Z is continuous and $Z_\infty = 0$ and let Λ be the increasing predictable process such that $M_t = H_t - \Lambda_{t \wedge \tau}$ is a martingale. Then the following conclusions hold:*

(i) the variable Λ_τ has standard exponential law (with parameter 1);

(ii) if $\mathbb{F}$ is immersed in $\mathbb{G}$, then the variable Λ_τ is independent of $\mathcal{F}_\infty$.

PROOF: (i) Fix $z > 0$ and consider the process $X = (X_t, t \geq 0)$, defined by:

$$X_t = (1+z)^{H_t}\, e^{-z\Lambda_{t\wedge\tau}}$$

for all $t \geq 0$. Then, applying the integration by parts formula, we get:

$$dX_t = z\, e^{-z\Lambda_{t\wedge\tau}}\, dM_t\,. \tag{3.2.2}$$

Hence, by virtue of the assumption that $z > 0$, it follows from (3.2.2) that X is a $\mathbb{G}$-martingale, so that:

$$\mathbb{E}\big[(1+z)^{H_t}\, e^{-z\Lambda_{t\wedge\tau}}\,\big|\,\mathcal{G}_s\big] = (1+z)^{H_s}\, e^{-z\Lambda_{s\wedge\tau}} \tag{3.2.3}$$

holds for all $0 \leq s \leq t$. (Note that the martingale property of X follows also from Exercise 2.1.10 for $h \equiv 1$.) In view of the implied by $z > 0$ uniform integrability of X, we may let t go to infinity in (3.2.3). Setting s equal to zero in (3.2.3), we therefore obtain:

$$\mathbb{E}\big[(1+z)\, e^{-z\Lambda_\tau}\big] = 1\,.$$

This means that the Laplace transform of Λ_τ is the same as one of a standard exponential variable and thus proves the claim.

(ii) Recall that, under immersion property, Z is decreasing and $d\Lambda = dZ/Z$. Applying the change-of-variable formula, we get, for continuous Z:

$$e^{-z\Lambda_{t\wedge\tau}} = 1 + z\int_0^t e^{-z\Lambda_s}\, \frac{1\!\!1_{(\tau>s)}}{Z_s}\, dZ_s \tag{3.2.4}$$

for all $t \geq 0$ and any $z > 0$ fixed. Then, taking conditional expectations under $\mathcal{F}_t$ from both parts of expression (3.2.4) and applying Fubini's theorem, we obtain from the immersion of $\mathbb{F}$ in $\mathbb{G}$ that:

$$\begin{aligned}
\mathbb{E}\big[e^{-z\Lambda_{t\wedge\tau}}\,\big|\,\mathcal{F}_t\big] &= 1 + z\int_0^t \mathbb{E}\left[e^{-z\Lambda_s}\, \frac{1\!\!1_{(\tau>s)}}{Z_s}\,\bigg|\,\mathcal{F}_t\right] dZ_s \\
&= 1 + z\int_0^t e^{-z\Lambda_s}\, \frac{\mathbb{P}(\tau > s\,|\,\mathcal{F}_t)}{Z_s}\, dZ_s \\
&= 1 + z\int_0^t e^{-z\Lambda_s}\, dZ_s
\end{aligned} \tag{3.2.5}$$

for all $t \geq 0$. Hence, using the fact that $\Lambda_t = -\ln Z_t$, we see from (3.2.5) that:

$$\mathbb{E}\big[e^{-z\Lambda_{t\wedge\tau}}\,|\,\mathcal{F}_t\big] = 1 + \frac{z}{1+z}\big((Z_t)^{1+z} - (Z_0)^{1+z}\big)$$

holds for all $t \geq 0$. Letting t go to infinity and using the assumption $Z_0 = 1$, as well as the fact that $Z_\infty = 0$ ($\mathbb{P}$-a.s.), we therefore obtain:

$$\mathbb{E}\big[e^{-z\Lambda_\tau}\,|\,\mathcal{F}_\infty\big] = \frac{1}{1+z}$$

that signifies the desired assertion. □

Exercise 3.2.7 (A different proof of Norros' result) Suppose that

$$\mathbb{P}(\tau \leq t|\mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where Γ is an arbitrary continuous strictly increasing $\mathbb{F}$-adapted process. Prove, using the inverse of Γ that the random variable Γ_τ is independent of $\mathcal{F}_\infty$, with exponential law of parameter 1. ◁

3.2.3 $\mathbb{G}$-martingales versus $\mathbb{F}$ martingales

Proposition 3.2.8 *Assume that $\mathbb{F}$ is immersed in $\mathbb{G}$. Let $Y^{\mathbb{G}}$ be a $\mathbb{G}$-adapted, $\mathbb{P}$-integrable process given by the formula*

$$Y_t^{\mathbb{G}} = y_t 1\!\!1_{\tau>t} + y_t(\tau)1\!\!1_{\tau\le t}, \quad \forall\, t \in \mathbb{R}_+, \tag{3.2.6}$$

where:
(i) the projection of $Y^{\mathbb{G}}$ onto $\mathbb{F}$, which is defined by

$$Y_t^{\mathbb{F}} := \mathbb{E}(Y_t^{\mathbb{G}}|\mathcal{F}_t) = y_t\,\mathbb{P}(\tau > t|\mathcal{F}_t) + \mathbb{E}(y_t(\tau)1\!\!1_{\tau\le t}|\mathcal{F}_t),$$

is a $(\mathbb{P},\mathbb{F})$-martingale,
(ii) for any fixed $u \in \mathbb{R}_+$, the process $(y_t(u),\, t \in [u,\infty))$ is a $(\mathbb{P},\mathbb{F})$-martingale.
Then the process $Y^{\mathbb{G}}$ is a $(\mathbb{P},\mathbb{G})$-martingale.

Proof: Let us take $s < t$. Then

$$\begin{aligned}\mathbb{E}(Y_t^{\mathbb{G}}|\mathcal{G}_s) &= \mathbb{E}(y_t 1\!\!1_{\tau>t}|\mathcal{G}_s) + \mathbb{E}(y_t(\tau)1\!\!1_{s<\tau\le t}|\mathcal{G}_s) + \mathbb{E}(y_t(\tau)1\!\!1_{\tau\le s}|\mathcal{G}_s)\\ &= 1\!\!1_{s<\tau}\frac{1}{Z_s}(\mathbb{E}(y_t Z_t|\mathcal{F}_s) + \mathbb{E}(y_t(\tau)1\!\!1_{s<\tau\le t}|\mathcal{F}_s)) + \mathbb{E}(y_t(\tau)1\!\!1_{\tau\le s}|\mathcal{G}_s)\end{aligned}$$

On the one hand,

$$\mathbb{E}(y_t(\tau)1\!\!1_{\tau\le s}|\mathcal{G}_s) = 1\!\!1_{\tau\le s}y_s(\tau) \tag{3.2.7}$$

Indeed, it suffices to prove the previous equality for $y_t(u) = h(u)X_t$ where X is an $\mathbb{F}$-martingale. In that case,

$$\mathbb{E}(X_t h(\tau)1\!\!1_{\tau\le s}|\mathcal{G}_s) = 1\!\!1_{\tau\le s}h(\tau)\mathbb{E}_{\mathbb{P}}(X_t|\mathcal{G}_s) = 1\!\!1_{\tau\le s}h(\tau)\mathbb{E}(X_t|\mathcal{F}_s) = 1\!\!1_{\tau\le s}h(\tau)X_s = 1\!\!1_{\tau\le s}y_s(\tau)$$

In the other hand, from (i)

$$\mathbb{E}(y_t Z_t + y_t(\tau)1\!\!1_{\tau\le t}|\mathcal{F}_s) = y_s Z_s + \mathbb{E}(y_s(\tau)1\!\!1_{\tau\le s}|\mathcal{F}_s)$$

It follows that

$$\mathbb{E}(Y_t^{\mathbb{G}}|\mathcal{G}_s) = 1\!\!1_{s<\tau}\frac{1}{Z_s}(y_s Z_s + \mathbb{E}((y_s(\tau) - y_t(\tau))1\!\!1_{\tau\le s}|\mathcal{F}_s)) + 1\!\!1_{\tau\le s}y_s(\tau)$$

It remains to check that

$$\mathbb{E}((y_s(\tau) - y_t(\tau))1\!\!1_{\tau\le s}|\mathcal{F}_s) = 0$$

which follows from

$$\mathbb{E}(y_t(\tau)1\!\!1_{\tau\le s}|\mathcal{F}_s) = \mathbb{E}(y_t(\tau)1\!\!1_{\tau\le s}|\mathcal{G}_s|\mathcal{F}_s) = \mathbb{E}(y_s(\tau)1\!\!1_{\tau\le s}|\mathcal{F}_s)$$

where we have used (3.2.7). □

Exercise 3.2.9 In a Cox model, for a continuous Λ prove that τ is independent of $\mathcal{F}_\infty$ if and only if λ is a deterministic function. ◁

Exercise 3.2.10 Prove that, if $\mathbb{P}(\tau > t|\mathcal{F}_t)$ is continuous and strictly decreasing, then there exists Θ independent of $\mathcal{F}_\infty$ such that $\tau = \inf\{t \,:\, \Lambda_t > \Theta\}$. ◁

Exercise 3.2.11 In a Cox model, write the Doob-Meyer and the multiplicative decomposition of Z. ◁

Exercise 3.2.12 Show how one can compute $\mathbb{P}(\tau > t|\mathcal{F}_t)$ when

$$\tau = \inf\{t \,:\, X_t > \Theta\}$$

where X is an $\mathbb{F}$-adapted process, not necessarily increasing, and Θ independent of $\mathcal{F}_\infty$. Does immersion property still holds? Same questions if Θ is not independent of $\mathcal{F}_\infty$. ◁

3.2.4 Martingale Representation Theorems

Theorem 3.2.13 *Suppose that $\mathbb{F}$ is immersed in $\mathbb{G}$ and that any $\mathbb{F}$-martingale is continuous. Then the martingale $M^h_t = \mathbb{E}(h_\tau \,|\, \mathcal{G}_t)$, where h is an $\mathbb{F}$-predictable process such that $\mathbb{E}|h_\tau| < \infty$, admits the following decomposition in the sum of a continuous martingale and a discontinuous martingale*

$$M^h_t := m^h_0 + \int_0^{t\wedge\tau} \frac{1}{Z_u}\, dm^h_u + \int_{]0,t\wedge\tau]} (h_u - M^h_{u-})\, dM_u, \tag{3.2.8}$$

where m^h is the continuous $\mathbb{F}$-martingale given by

$$m^h_t := -\mathbb{E}\Big(\int_0^\infty h_u\, dZ_u \,\Big|\, \mathcal{F}_t \Big)$$

and M is the discontinuous $\mathbb{G}$-martingale defined as $M_t = H_t - \Gamma_{t\wedge\tau}$, where $\Gamma = -\ln Z$.

PROOF: We start by noting that

$$\begin{aligned} M^h_t &= \mathbb{E}(h_\tau|\mathcal{G}_t) = 1\!\!1_{\{t\geq\tau\}} h_\tau - 1\!\!1_{\{t<\tau\}} e^{\Gamma_t}\, \mathbb{E}\Big(\int_t^\infty h_u\, dZ_u \,\Big|\, \mathcal{F}_t \Big) \\ &= 1\!\!1_{\{t\geq\tau\}} h_\tau + 1\!\!1_{\{t<\tau\}} e^{\Gamma_t} \Big(m^h_t + \int_0^t h_u\, dZ_u \Big). \end{aligned} \tag{3.2.9}$$

We will sketch two slightly different derivations of (3.2.8).

First derivation. Let the process J be given by the formula, for $t \in \mathbb{R}_+$,

$$J_t = e^{\Gamma_t} \Big(m^h_t + \int_0^t h_u\, dZ_u \Big).$$

Noting that Γ is a continuous increasing process and m^h is a continuous martingale, we deduce from the Itô integration by parts formula that

$$\begin{aligned} dJ_t &:= e^{\Gamma_t}\, dm^h_t - e^{\Gamma_t} h_t\, dF_t + \Big(m^h_t + \int_0^t h_u\, dZ_u \Big) e^{\Gamma_t}\, d\Gamma_t \\ &:= e^{\Gamma_t}\, dm^h_t + e^{\Gamma_t} h_t\, dZ_t + J_t\, d\Gamma_t. \end{aligned}$$

Therefore, from $dZ_t = -e^{-\Gamma_t} d\Gamma_t$,

$$dJ_t = e^{\Gamma_t}\, dm^h_t + (J_t - h_t)\, d\Gamma_t$$

or, in the integrated form,

$$J_t = M^h_0 + \int_0^t e^{\Gamma_u}\, dm^h_u + \int_0^t (J_u - h_u)\, d\Gamma_u.$$

Note that $J_t = M^h_t = M^h_{t-}$ on the event $\{t < \tau\}$. Therefore, on the event $\{t < \tau\}$,

$$M^h_t = M^h_0 + \int_0^{t\wedge\tau} e^{\Gamma_u}\, dm^h_u + \int_0^{t\wedge\tau} (M^h_{u-} - h_u)\, d\Gamma_u.$$

From (3.2.9), the jump of M^h at time τ equals

$$h_\tau - J_\tau = h_\tau - M^h_{\tau-} = M^h_\tau - M^h_{\tau-}.$$

Equality (3.2.8) now easily follows.

Second derivation. Equality (3.2.9) can be re-written as follows

$$M_t^h = \int_0^t h_u\, dH_u + (1 - H_t)e^{\Gamma_t}\Big(m_t^h - \int_0^t h_u\, dF_u\Big).$$

Hence formula (3.2.8) can be obtained directly by the integration by parts formula. □

✓ Add Kusuoka's result, Le Cam's result, General Case (CJN) Add Bremaud Yor results on equality of SI in the two filtrations, and projection of $\int \varphi_s dB_s$ on $\mathbb{F}$

3.2.5 Immersion and Change of probability

Let us first examine a general set-up in which $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where $\mathbb{F}$ is an arbitrary filtration and $\mathbb{H}$ is generated by the default process H. We say that $\mathbb{Q}$ is locally equivalent to $\mathbb{P}$ if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_t)$ for every $t \in \mathbb{R}_+$. Then there exists the Radon-Nikodým density process L such that

$$d\mathbb{Q}\,|_{\,\mathcal{G}_t} = L_t\, d\mathbb{P}\,|_{\,\mathcal{G}_t}, \quad \forall\, t \in \mathbb{R}_+. \tag{3.2.10}$$

Part (i) in the next lemma is well known (see Jamshidian [66]). We assume that the immersion holds under $\mathbb{P}$.

Lemma 3.2.14 *(i) Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G}_t)$ for every $t \in \mathbb{R}_+$, with the associated Radon-Nikodým density process L. If the density process L is $\mathbb{F}$-adapted then we have $\mathbb{Q}(\tau \le t\,|\,\mathcal{F}_t) = \mathbb{P}(\tau \le t\,|\,\mathcal{F}_t)$ for every $t \in \mathbb{R}_+$. Hence, immersion is also valid under $\mathbb{Q}$ and the $\mathbb{F}$-intensities of τ under $\mathbb{Q}$ and under $\mathbb{P}$ coincide.*
(ii) Assume that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$ and $d\mathbb{Q} = L_\infty\, d\mathbb{P}$, so that $L_t = \mathbb{E}_{\mathbb{P}}(L_\infty\,|\,\mathcal{G}_t)$. Then immersion is valid under $\mathbb{Q}$ whenever we have, for every $t \in \mathbb{R}_+$,

$$\frac{\mathbb{E}_{\mathbb{P}}(L_\infty H_t\,|\,\mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(L_\infty\,|\,\mathcal{F}_\infty)} = \frac{\mathbb{E}_{\mathbb{P}}(L_t H_t\,|\,\mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(L_t\,|\,\mathcal{F}_\infty)}. \tag{3.2.11}$$

Proof: To prove (i), assume that the density process L is $\mathbb{F}$-adapted. We have for each $t \le s \in \mathbb{R}_+$

$$\mathbb{Q}(\tau \le t\,|\,\mathcal{F}_t) = \frac{\mathbb{E}_{\mathbb{P}}(L_t 1\!\!1_{\{\tau \le t\}}\,|\,\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(L_t\,|\,\mathcal{F}_t)} = \mathbb{P}(\tau \le t\,|\,\mathcal{F}_t) = \mathbb{P}(\tau \le t\,|\,\mathcal{F}_s) = \mathbb{Q}(\tau \le t\,|\,\mathcal{F}_s),$$

where the last equality follows by another application of the Bayes formula. The assertion follows.

To prove part (ii), it suffices to establish the equality

$$\widehat{F}_t := \mathbb{Q}(\tau \le t\,|\,\mathcal{F}_t) = \mathbb{Q}(\tau \le t\,|\,\mathcal{F}_\infty), \quad \forall\, t \in \mathbb{R}_+. \tag{3.2.12}$$

Note that since the random variables $L_t 1\!\!1_{\{\tau \le t\}}$ and L_t are $\mathbb{P}$-integrable and $\mathcal{G}_t$-measurable, using the Bayes formula, immersion hypothesis, and assumed equality (3.2.11), we obtain the following chain of equalities

$$\begin{aligned}\mathbb{Q}(\tau \le t\,|\,\mathcal{F}_t) &= \frac{\mathbb{E}_{\mathbb{P}}(L_t 1\!\!1_{\{\tau \le t\}}\,|\,\mathcal{F}_t)}{\mathbb{E}_{\mathbb{P}}(L_t\,|\,\mathcal{F}_t)} = \frac{\mathbb{E}_{\mathbb{P}}(L_t 1\!\!1_{\{\tau \le t\}}\,|\,\mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(L_t\,|\,\mathcal{F}_\infty)} \\ &= \frac{\mathbb{E}_{\mathbb{P}}(L_\infty 1\!\!1_{\{\tau \le t\}}\,|\,\mathcal{F}_\infty)}{\mathbb{E}_{\mathbb{P}}(L_\infty\,|\,\mathcal{F}_\infty)} = \mathbb{Q}(\tau \le t\,|\,\mathcal{F}_\infty).\end{aligned}$$

We conclude that immersion holds under $\mathbb{Q}$ if and only if (3.2.11) is valid. □

Unfortunately, straightforward verification of condition (3.1.1) is rather cumbersome. For this reason, we shall provide alternative sufficient conditions for the preservation of immersion under a locally equivalent probability measure.

Case of the Brownian filtration

Let W be a Brownian motion under $\mathbb{P}$ and $\mathbb{F}$ its natural filtration. Since we work under immersion hypothesis, the process W is also a $\mathbb{G}$-martingale, where $\mathbb{G} = \mathbb{F}\vee\mathbb{H}$. Hence, W is a Brownian motion with respect to $\mathbb{G}$ under $\mathbb{P}$. Our goal is to show that immersion is still valid under $\mathbb{Q} \in \mathcal{Q}$ for a large class $\mathcal{Q}$ of (locally) equivalent probability measures on $(\Omega, \mathcal{G})$.

Let $\mathbb{Q}$ be an arbitrary probability measure locally equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$. Kusuoka [88] (see also Section 5.2.2 in Bielecki and Rutkowski [23]) proved that, under immersion hypothesis, any $\mathbb{G}$-martingale under $\mathbb{P}$ can be represented as the sum of stochastic integrals with respect to the Brownian motion W and the jump martingale M. In our set-up, Kusuoka's representation theorem implies that there exist $\mathbb{G}$-predictable processes θ and $\zeta > -1$, such that the Radon-Nikodým density L of $\mathbb{Q}$ with respect to $\mathbb{P}$ satisfies the following SDE

$$dL_t = L_{t-}\big(\theta_t\, dW_t + \zeta_t\, dM_t\big) \tag{3.2.13}$$

with the initial value $L_0 = 1$. More explicitly, the process η equals

$$L_t = \mathcal{E}_t\left(\int_0^\cdot \theta_u\, dW_u\right)\mathcal{E}_t\left(\int_0^\cdot \zeta_u\, dM_u\right) = L_t^{(1)}L_t^{(2)}, \tag{3.2.14}$$

where we write

$$L_t^{(1)} = \mathcal{E}_t\left(\int_0^\cdot \theta_u\, dW_u\right) = \exp\left(\int_0^t \theta_u\, dW_u - \frac{1}{2}\int_0^t \theta_u^2\, du\right), \tag{3.2.15}$$

and

$$L_t^{(2)} = \mathcal{E}_t\left(\int_0^\cdot \zeta_u\, dM_u\right) = \exp\left(\int_0^t \ln(1+\zeta_u)\, dH_u - \int_0^{t\wedge\tau} \zeta_u\gamma_u\, du\right). \tag{3.2.16}$$

Moreover, by virtue of a suitable version of Girsanov's theorem, the following processes $\widehat{W}$ and $\widehat{M}$ are $\mathbb{G}$-martingales under $\mathbb{Q}$

$$\widehat{W}_t = W_t - \int_0^t \theta_u\, du, \quad \widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u<\tau\}}\gamma_u\zeta_u\, du. \tag{3.2.17}$$

Proposition 3.2.15 *Assume that immersion holds under $\mathbb{P}$. Let $\mathbb{Q}$ be a probability measure locally equivalent to $\mathbb{P}$ with the associated Radon-Nikodým density process L given by formula (3.2.14) . If the process θ is $\mathbb{F}$-adapted then immersion is valid under $\mathbb{Q}$ and the $\mathbb{F}$-intensity of τ under $\mathbb{Q}$ equals $\widehat{\gamma}_t = (1+\widetilde{\zeta}_t)\gamma_t$, where $\widetilde{\zeta}$ is the unique $\mathbb{F}$-predictable process such that the equality $\widetilde{\zeta}_t\mathbb{1}_{\{t\le\tau\}} = \zeta_t\mathbb{1}_{\{t\le\tau\}}$ holds for every $t \in \mathbb{R}_+$.*

PROOF: Let $\mathbb{P}^*$ be the probability measure locally equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{G})$, given by

$$d\mathbb{P}^*\,|\,_{\mathcal{G}_t} = \mathcal{E}_t\left(\int_0^\cdot \zeta_u\, dM_u\right) d\mathbb{P}\,|\,_{\mathcal{G}_t} = L_t^{(2)}\, d\mathbb{P}\,|\,_{\mathcal{G}_t}. \tag{3.2.18}$$

We claim that immersion holds under $\mathbb{P}^*$. From Girsanov's theorem, the process W follows a Brownian motion under $\mathbb{P}^*$ with respect to both $\mathbb{F}$ and $\mathbb{G}$. Moreover, from the predictable representation property of W under $\mathbb{P}^*$, we deduce that any $\mathbb{F}$-local martingale L under $\mathbb{P}^*$ can be written as a stochastic integral with respect to W. Specifically, there exists an $\mathbb{F}$-predictable process ξ such that

$$L_t = L_0 + \int_0^t \xi_u\, dW_u.$$

This shows that L is also a $\mathbb{G}$-local martingale, and thus immersion holds under $\mathbb{P}^*$. Since

$$d\mathbb{Q}\,|\,_{\mathcal{G}_t} = \mathcal{E}_t\left(\int_0^\cdot \theta_u\, dW_u\right) d\mathbb{P}^*\,|\,_{\mathcal{G}_t},$$

by virtue of part (i) in Lemma 3.2.14, immersion is valid under $\mathbb{Q}$ as well. The last claim in the statement of the lemma can be deduced from the fact that immersion holds under $\mathbb{Q}$ and, by Girsanov's theorem, the process

$$\widehat{M}_t = M_t - \int_0^t \mathbb{1}_{\{u<\tau\}}\gamma_u\zeta_u\,du = H_t - \int_0^t \mathbb{1}_{\{u<\tau\}}(1+\widetilde{\zeta}_u)\gamma_u\,du$$

is a $\mathbb{Q}$-martingale. □

We claim that the equality $\mathbb{P}^* = \mathbb{P}$ holds on the filtration $\mathbb{F}$. Indeed, we have $d\mathbb{P}^*\,|\,_{\mathcal{F}_t} = \widetilde{L}_t\,d\mathbb{P}\,|\,_{\mathcal{F}_t}$, where we write $\widetilde{L}_t = \mathbb{E}_{\mathbb{P}}(L_t^{(2)}\,|\,\mathcal{F}_t)$, and

$$\mathbb{E}_{\mathbb{P}}(L_t^{(2)}\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\left(\int_0^{\cdot}\zeta_u\,dM_u\right)\Big|\,\mathcal{F}_\infty\right) = 1, \quad \forall\, t\in\mathbb{R}_+, \tag{3.2.19}$$

where the first equality follows immersion.

To establish the second equality in (3.2.19), we first note that since the process M is stopped at τ, we may assume, without loss of generality, that $\zeta = \widetilde{\zeta}$ where the process $\widetilde{\zeta}$ is $\mathbb{F}$-predictable. Moreover,the conditional cumulative distribution function of τ given $\mathcal{F}_\infty$ has the form $1-\exp(-\Gamma_t(\omega))$. Hence, for arbitrarily selected sample paths of processes ζ and Γ, the claimed equality can be seen as a consequence of the martingale property of the Doléans exponential.

Formally, it can be proved by following elementary calculations, where the first equality is a consequence of (3.2.16)),

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\left(\int_0^{\cdot}\widetilde{\zeta}_u\,dM_u\right)\Big|\,\mathcal{F}_\infty\right) &= \mathbb{E}_{\mathbb{P}}\left(\big(1+\mathbb{1}_{\{t\geq\tau\}}\widetilde{\zeta}_\tau\big)\exp\Big(-\int_0^{t\wedge\tau}\widetilde{\zeta}_u\gamma_u\,du\Big)\Big|\,\mathcal{F}_\infty\right)\\
&= \mathbb{E}_{\mathbb{P}}\left(\int_0^\infty\big(1+\mathbb{1}_{\{t\geq u\}}\widetilde{\zeta}_u\big)\exp\Big(-\int_0^{t\wedge u}\widetilde{\zeta}_v\gamma_v\,dv\Big)\gamma_u e^{-\int_0^u\gamma_v\,dv}du\,\Big|\,\mathcal{F}_\infty\right)\\
&= \mathbb{E}_{\mathbb{P}}\left(\int_0^t(1+\widetilde{\zeta}_u)\gamma_u\exp\Big(-\int_0^u(1+\widetilde{\zeta}_v)\gamma_v\,dv\Big)du\,\Big|\,\mathcal{F}_\infty\right)\\
&+\exp\Big(-\int_0^t\widetilde{\zeta}_v\gamma_v\,dv\Big)\,\mathbb{E}_{\mathbb{P}}\left(\int_t^\infty\gamma_u e^{-\int_0^u\gamma_v\,dv}du\,\Big|\,\mathcal{F}_\infty\right)\\
&= \int_0^t(1+\widetilde{\zeta}_u)\gamma_u\exp\Big(-\int_0^u(1+\widetilde{\zeta}_v)\gamma_v\,dv\Big)du\\
&+\exp\Big(-\int_0^t\widetilde{\zeta}_v\gamma_v\,dv\Big)\int_t^\infty\gamma_u e^{-\int_0^u\gamma_v\,dv}du\\
&= 1-\exp\Big(-\int_0^t(1+\widetilde{\zeta}_v)\gamma_v\,dv\Big)+\exp\Big(-\int_0^t\widetilde{\zeta}_v\gamma_v\,dv\Big)\exp\Big(-\int_0^t\gamma_v\,dv\Big) = 1,
\end{aligned}$$

where the second last equality follows by an application of the chain rule.

Extension to orthogonal martingales

Equality (3.2.19) suggests that Proposition 3.2.15 can be extended to the case of arbitrary orthogonal local martingales. Such a generalization is convenient, if we wish to cover the situation considered in Kusuoka's counterexample.

Let N be a local martingale under $\mathbb{P}$ with respect to the filtration $\mathbb{F}$. It is also a $\mathbb{G}$-local martingale, since we maintain the assumption that immersion holds under $\mathbb{P}$. Let $\mathbb{Q}$ be an arbitrary probability measure locally equivalent to $\mathbb{P}$ on $(\Omega,\mathcal{G})$. We assume that the Radon-Nikodým density process L of $\mathbb{Q}$ with respect to $\mathbb{P}$ equals

$$dL_t = L_{t-}\big(\theta_t\,dN_t + \zeta_t\,dM_t\big) \tag{3.2.20}$$

for some $\mathbb{G}$-predictable processes θ and $\zeta > -1$ (the properties of the process θ depend, of course, on the choice of the local martingale N). The next result covers the case where N and M are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$, so that the product MN follows a $\mathbb{G}$-local martingale.

Proposition 3.2.16 *Assume that the following conditions hold:*
(a) N and M are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$,
(b) N has the predictable representation property under $\mathbb{P}$ with respect to $\mathbb{F}$, in the sense that any $\mathbb{F}$-local martingale L under $\mathbb{P}$ can be written as

$$L_t = L_0 + \int_0^t \xi_u \, dN_u, \quad \forall \, t \in \mathbb{R}_+,$$

for some $\mathbb{F}$-predictable process ξ,
(c) $\mathbb{P}^*$ is a probability measure on $(\Omega, \mathcal{G})$ such that (3.2.18) holds.
Then we have:
(i) immersion is valid under $\mathbb{P}^*$,
(ii) if the process θ is $\mathbb{F}$-adapted then immersion is valid under $\mathbb{Q}$.

The proof of the proposition hinges on the following simple lemma.

Lemma 3.2.17 *Under the assumptions of Proposition 3.2.16, we have:*
(i) N is a $\mathbb{G}$-local martingale under $\mathbb{P}^*$,
(ii) N has the predictable representation property for $\mathbb{F}$-local martingales under $\mathbb{P}^*$.

Proof: In view of (c), we have $d\mathbb{P}^* \,|\, _{\mathcal{G}_t} = L_t^{(2)} \, d\mathbb{P} \,|\, _{\mathcal{G}_t}$, where the density process $L^{(2)}$ is given by (3.2.16), so that $dL_t^{(2)} = L_{t-}^{(2)} \zeta_t \, dM_t$. From the assumed orthogonality of N and M, it follows that N and $L^{(2)}$ are orthogonal $\mathbb{G}$-local martingales under $\mathbb{P}$, and thus $NL^{(2)}$ is a $\mathbb{G}$-local martingale under $\mathbb{P}$ as well. This means that N is a $\mathbb{G}$-local martingale under $\mathbb{P}^*$, so that (i) holds.

To establish part (ii) in the lemma, we first define the auxiliary process $\widetilde{L}$ by setting $\widetilde{L}_t = \mathbb{E}_{\mathbb{P}}(L_t^{(2)} \,|\, \mathcal{F}_t)$. Then manifestly $d\mathbb{P}^* \,|\, _{\mathcal{F}_t} = \widetilde{L}_t \, d\mathbb{P} \,|\, _{\mathcal{F}_t}$, and thus in order to show that any $\mathbb{F}$-local martingale under $\mathbb{P}^*$ follows an $\mathbb{F}$-local martingale under $\mathbb{P}$, it suffices to check that $\widetilde{\eta}_t = 1$ for every $t \in \mathbb{R}_+$, so that $\mathbb{P}^* = \mathbb{P}$ on $\mathbb{F}$. To this end, we note that

$$\mathbb{E}_{\mathbb{P}}(L_t^{(2)} \,|\, \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}} \left(\mathcal{E}_t \left(\int_0^{\cdot} \zeta_u \, dM_u \right) \Big| \, \mathcal{F}_\infty \right) = 1, \quad \forall \, t \in \mathbb{R}_+,$$

where the first equality follows from immersion property, and the second one can established similarly as the second equality in (3.2.19).

We are in a position to prove (ii). Let L be an $\mathbb{F}$-local martingale under $\mathbb{P}^*$. Then it follows also an $\mathbb{F}$-local martingale under $\mathbb{P}$ and thus, by virtue of (b), it admits an integral representation with respect to N under $\mathbb{P}$ and $\mathbb{P}^*$. This shows that N has the predictable representation property with respect to $\mathbb{F}$ under $\mathbb{P}^*$. □

We now proceed to the proof of Proposition 3.2.16.

Proof of Proposition 3.2.16. We shall argue along the similar lines as in the proof of Proposition 3.2.15. To prove (i), note that by part (ii) in Lemma 3.2.17 we know that any $\mathbb{F}$-local martingale under $\mathbb{P}^*$ admits the integral representation with respect to N. But, by part (i) in Lemma 3.2.17, N is a $\mathbb{G}$-local martingale under $\mathbb{P}^*$. We conclude that L is a $\mathbb{G}$-local martingale under $\mathbb{P}^*$, and thus the immersion is valid under $\mathbb{P}^*$. Assertion (ii) now follows from part (i) in Lemma 3.2.14. □

Remark 3.2.18 It should be stressed that Proposition 3.2.16 is not directly employed in what follows. We decided to present it here, since it sheds some light on specific technical problems arising in the context of modeling dependent default times through an equivalent change of a probability measure (see Kusuoka [88]).

Example 3.2.19 Kusuoka [88] presents a counter-example based on the two independent random times τ_1 and τ_2 given on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$. We write $M_t^i = H_t^i - \int_0^{t\wedge\tau_i} \gamma_i(u)\,du$, where $H_t^i = \mathbb{1}_{\{t\geq\tau_i\}}$ and γ_i is the deterministic intensity function of τ_i under $\mathbb{P}$. Let us set $d\mathbb{Q}\,|\,_{\mathcal{G}_t} = L_t\, d\mathbb{P}\,|\,_{\mathcal{G}_t}$, where $L_t = L_t^{(1)} L_t^{(2)}$ and, for $i = 1, 2$ and every $t \in \mathbb{R}_+$,

$$L_t^{(i)} = 1 + \int_0^t L_{u-}^{(i)} \zeta_u^{(i)}\, dM_u^i = \mathcal{E}_t\left(\int_0^{\cdot} \zeta_u^{(i)}\, dM_u^i\right)$$

for some $\mathbb{G}$-predictable processes $\zeta^{(i)}$, $i = 1, 2$, where $\mathbb{G} = \mathbb{H}^1 \vee \mathbb{H}^2$. We set $\mathbb{F} = \mathbb{H}^1$ and $\mathbb{H} = \mathbb{H}^2$. Manifestly, the immersion holds under $\mathbb{P}$. Moreover, in view of Proposition 3.2.16, it is still valid under the equivalent probability measure $\mathbb{P}^*$ given by

$$d\mathbb{P}^*\,|\,_{\mathcal{G}_t} = \mathcal{E}_t\left(\int_0^{\cdot} \zeta_u^{(2)}\, dM_u^2\right) d\mathbb{P}\,|\,_{\mathcal{G}_t}.$$

It is clear that $\mathbb{P}^* = \mathbb{P}$ on $\mathbb{F}$, since

$$\mathbb{E}_{\mathbb{P}}(L_t^{(2)}\,|\,\mathcal{F}_t) = \mathbb{E}_{\mathbb{P}}\left(\mathcal{E}_t\left(\int_0^{\cdot} \zeta_u^{(2)}\, dM_u^2\right)\,\Big|\,\mathcal{H}_t^1\right) = 1, \quad \forall\, t \in \mathbb{R}_+.$$

However,immersion is not necessarily valid under $\mathbb{Q}$ if the process $\zeta^{(1)}$ fails to be $\mathbb{F}$-adapted. In Kusuoka's counter-example, the process $\zeta^{(1)}$ was chosen to be explicitly dependent on both random times, and it was shown that immersiondoes not hold under $\mathbb{Q}$. For an alternative approach to Kusuoka's example, through an absolutely continuous change of a probability measure, the interested reader may consult Collin-Dufresne *et al.* [30].

3.3 Successive Enlargements

3.3.1 Immersion

Proposition 3.3.1 *Let $\tau_1 < \tau_2$ a.s., $\mathbb{H}^i$ be the filtration generated by the default process $H_t^i = \mathbb{1}_{\tau_i \leq t}$, and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2$. Then, the two following assertions are equivalent:*
(i) $\mathbb{F}$ is immersed in $\mathbb{G}$
(ii) $\mathbb{F}$ is immersed in $\mathbb{F} \vee \mathbb{H}^1$ and $\mathbb{F} \vee \mathbb{H}^1$ is immersed in $\mathbb{G}$.

Proof: (this result was obtained by Ehlers and Schönbucher [41], we give here a slightly different proof.) The only fact to check is that if $\mathbb{F}$ is immersed in $\mathbb{G}$, then $\mathbb{F} \vee \mathbb{H}^1$ is immersed in $\mathbb{G}$, or that

$$\mathbb{P}(\tau_2 > t | \mathcal{F}_t \vee \mathcal{H}_t^1) = \mathbb{P}(\tau_2 > t | \mathcal{F}_\infty \vee \mathcal{H}_\infty^1)$$

This is equivalent to, for any h, and any $A_\infty \in \mathcal{F}_\infty$

$$\mathbb{E}(A_\infty h(\tau_1) \mathbb{1}_{\tau_2 > t}) = \mathbb{E}(A_\infty h(\tau_1) \mathbb{P}(\tau_2 > t | \mathcal{F}_t \vee \mathcal{H}_t^1))$$

We spilt this equality in two parts. The first equality

$$\mathbb{E}(A_\infty h(\tau_1) \mathbb{1}_{\tau_1 > t} \mathbb{1}_{\tau_2 > t}) = \mathbb{E}(A_\infty h(\tau_1) \mathbb{1}_{\tau_1 > t} \mathbb{P}(\tau_2 > t | \mathcal{F}_t \vee \mathcal{H}_t^1))$$

is obvious since $\mathbb{1}_{\tau_1 > t} \mathbb{1}_{\tau_2 > t} = \mathbb{1}_{\tau_1 > t}$ and $\mathbb{1}_{\tau_1 > t} \mathbb{P}(\tau_2 > t | \mathcal{F}_t \vee \mathcal{H}_t^1) = \mathbb{1}_{\tau_1 > t}$.

Since $\mathbb{F}$ is immersed in $\mathbb{G}$, one has $\mathbb{E}(A_\infty | \mathcal{G}_t) = \mathbb{E}(A_\infty | \mathcal{F}_t)$ and it follows (WHY?) that $\mathbb{E}(A_\infty | \mathcal{G}_t) = \mathbb{E}(A_\infty | \mathcal{F}_t \vee \mathcal{H}_t^1)$, therefore

$$\begin{aligned}
\mathbb{E}(A_\infty h(\tau_1) \mathbb{1}_{\tau_2 > t \geq \tau_1}) &= \mathbb{E}(\mathbb{E}(A_\infty | \mathcal{G}_t) h(\tau_1) \mathbb{1}_{\tau_2 > t \geq \tau_1}) \\
&= \mathbb{E}(\mathbb{E}(A_\infty | \mathcal{F}_t \vee \mathcal{H}_t^1) h(\tau_1) \mathbb{1}_{\tau_2 > t \geq \tau_1}) \\
&= \mathbb{E}(\mathbb{E}(A_\infty | \mathcal{F}_t \vee \mathcal{H}_t^1) \mathbb{E}(h(\tau_1) \mathbb{1}_{\tau_2 > t \geq \tau_1} | \mathcal{F}_t \vee \mathcal{H}_t^1)) \\
&= \mathbb{E}(A_\infty \mathbb{E}(h(\tau_1) \mathbb{1}_{\tau_2 > t \geq \tau_1} | \mathcal{F}_t \vee \mathcal{H}_t^1))
\end{aligned}$$

Exercise 3.3.2 Prove that $\mathbb{H}^i, i = 1,2$ are immersed in $\mathbb{H}^1 \vee \mathbb{H}^2$ if and only if $\tau_i, i = 1,2$ are independent. $\triangleleft$

3.3.2 Various immersion

Lemma 3.3.3 *Let $\mathbb{F}$ be generated by a Brownian motion. Assume that $\mathbb{F}$ is immersed in $\mathbb{G}^1 = \mathbb{F} \vee \mathbb{H}^1$ and in $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \mathbb{H}^2$ and that there exists an $\mathbb{F}$ predictable increasing process Λ^1 such that $M^1_t = H^1_t - \Lambda^1_{t\wedge\tau_1}$ is a $\mathbb{G}$ martingale. Then $\mathbb{G}^1$ is immersed in $\mathbb{G}$*

Proof: Any $\mathbb{G}^1$ martingale admits a decomposition as $Y_t = y + \int_0^t y_s dW_s + \int_0^t \widetilde{y}_s dM^1_s$. The result follows since W and M^1 are assumed to be $\mathbb{G}$ martingales $\square$

This result extends to the case of an arbitrary filtration $\mathbb{F}$. Indeed, for $X \in b\mathcal{F}_T$ and h bounded Borel function

$$\mathbb{E}(Xh(\tau_1)|\mathcal{G}^1_t) = h(\tau_1)1\!\!1_{\tau_1\le t}\mathbb{E}(X|\mathcal{G}^1_t) + 1\!\!1_{t<\tau_1}\frac{1}{Z^1_t}\mathbb{E}(X\int_t^\infty h(u)dF_u|\mathcal{F}_t)$$

can be written as a sum of stochastic integrals wrt M^1 and to some $\mathbb{F}$ martingales (note that, from immersion $\mathbb{E}(X|\mathcal{G}^1_t) = \mathbb{E}(X|\mathcal{F}_t)$.

3.3.3 Norros' lemma

Lemma 3.3.4 *Norros Lemma.*
Let $\tau_i, i = 1, \cdots, n$ be n finite-valued random times and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}^1_t \vee \cdots \vee \mathcal{H}^n_t$. Assume that

(i) $P(\tau_i = \tau_j) = 0, \forall i \neq j$

(ii) there exists continuous increasing processes Λ^i such that $M^i_t = H^i_t - \Lambda^i_{t\wedge\tau_i}$ are $\mathbb{G}$-martingales

then, the r.v's $\Lambda^i_{\tau_i}$ are independent with exponential law.

Proof: For any $\mu_i > -1$, the processes $L^i_t = (1+\mu_i)^{H^i_t} e^{-\mu_i \Lambda^i_{t\wedge\tau_i}}$, solution of

$$dL^i_t = L^i_{t-}\mu_i dM^i_t$$

are uniformly integrable martingales. Moreover, these martingales have no common jumps, and are orthogonal. Hence $E(\prod_i (1+\mu_i)e^{-\mu_i\Lambda^i_{\tau_i}}) = 1$, which implies

$$E(\prod_i e^{-\mu_i\Lambda^i_{\tau_i}}) = \prod_i (1+\mu_i)^{-1}$$

hence the independence property. $\square$

Application: Let us study the particular case of Poisson process. Let τ_1 and τ_2 are the two first jumps of a Poisson process, we have

$$G(t,s) = \begin{cases} e^{-\lambda t} \text{ for } s < t \\ e^{-\lambda s}(1 + \lambda(s-t)) \text{ for } s > t \end{cases}$$

with partial derivatives

$$\partial_1 G(t,s) = \begin{cases} -\lambda e^{-\lambda t} \text{ for } t > s \\ -\lambda e^{-\lambda s} \text{ for } s > t \end{cases}, \quad \partial_2 G(t,s) = \begin{cases} 0 \text{ for } t > s \\ -\lambda^2 e^{-\lambda s}(s-t) \text{ for } s > t \end{cases}$$

and

$$h(t,s) = \begin{cases} 1 \text{ for } t > s \\ \frac{t}{s} \text{ for } s > t \end{cases}, \quad \partial_1 h(t,s) = \begin{cases} 0 \text{ for } t > s \\ \frac{1}{s} \text{ for } s > t \end{cases}$$

$$k(t,s) = \begin{cases} 0 \text{ for } t > s \\ 1 - e^{-\lambda(s-t)} \text{ for } s > t \end{cases}, \quad \partial_2 k(t,s) = \begin{cases} 0 \text{ for } t > s \\ \lambda e^{-\lambda(s-t)} \text{ for } s > t \end{cases}$$

Then, one obtains $\Lambda_{\tau_1} = \tau_1$ et $\Lambda_{\tau_2} = \tau_2 - \tau_1$

3.3.4 Several Defaults in a Cox model

Proposition 3.3.5 *Let $\tau_i := \inf\{t : \Lambda^i_t \geq \Theta_i\}$, where the Θ_i's are independent from $\mathbb{F}$ and Λ^i's are $\mathbb{F}$ adapted increasing processes. Let $\mathbb{H}^i$ be the natural filtration of H^i, where $H^i_t = 1\!\!1_{\tau_i \leq t}$ and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^n$ be the full observation filtration. Then $\mathbb{F}$ is immersed in $\mathbb{G}$.*

Proof: Observe that, $\mathbb{G} \subset \mathbb{F} \vee \sigma(\Theta^1) \vee \cdots \vee \sigma(\Theta^n)$ and that, from the independence hypothesis, obviously $\mathbb{F}$ is immersed in $\mathbb{F} \vee \sigma(\Theta^1) \vee \cdots \vee \sigma(\Theta^n)$. □

Corollary 3.3.6 *In the case where Θ^i are independent, $\mathbb{G}^i := \mathbb{F} \vee \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^i$ is immersed in $\mathbb{G}$ and the $\mathbb{G}^i$ intensity of τ_i is the $\mathbb{G}$ intensity. The filtration $\mathbb{F}^i := \mathbb{F} \vee \mathbb{H}^i$ is immersed in $\mathbb{G}$ and the $\mathbb{F}^i$ intensity of τ_i is the $(\mathbb{F}, \mathbb{G})$ intensity.*

Proof: $\mathbb{F} \vee \mathbb{H}^1 \vee \cdots \vee \mathbb{H}^i$ is immersed in $\mathbb{F} \vee \sigma(\Theta^1) \vee \cdots \vee \sigma(\Theta^n)$. □

It is important to note that in the case of Proposition 3.3.5, the $(\mathbb{F}, \mathbb{G}^i)$ intensity of τ_i is not equal to its $(\mathbb{F}, \mathbb{G})$ intensity. In other terms, $\mathbb{G}^i$ is not immersed in $\mathbb{G}$ in that general setting.

We can extend the characterization of Cox model with immersion property as follows. We keep the notation of the previous Proposition.

Proposition 3.3.7 *We assume that $\mathbb{P}(\tau_i = \tau_j) = 0$ for $i \neq j$. If, for any $i = 0, \ldots, n$, $\mathbb{G}^i$ is immersed in $\mathbb{G}$ and if there exists $\mathbb{F}$-adapted processes Λ^i such that $M^i_t := H^i_t - \Lambda^i_{t \wedge \tau_i}$ are $\mathbb{G}^i$ martingales, then, there exist independent r.vs Θ^i, independent from $\mathbb{F}$ such that $\tau_i = \inf\{t : \Lambda^i_t \geq \Theta^i\}$.*

Proof: The fact that $\Theta^i := \Lambda_{\tau_i}$ are independent follows from Norros'lemma. The Θ^i are independent from $\mathbb{F}$ from the single default case. Note that our hypothesis implies that M^i are $\mathbb{F} \vee \mathbb{H}^i$ martingales and $\mathbb{G}$ martingales and that, from Corollary 3.3.6, $\mathbb{F} \vee \mathbb{H}^i$ is immersed in $\mathbb{G}$. □

3.3.5 Kusuoka counter example

Kusuoka [88] presents a counter-example of the stability of $\mathcal{H}$ hypothesis under a change of probability, based on two independent random times τ_1 and τ_2 given on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$ and admitting a density w.r.t. Lebesgue's measure. The process $M^1_t = H^1_t - \int_0^{t \wedge \tau_1} \lambda_1(u)\, du$, where $H^1_t = 1\!\!1_{\{t \geq \tau_1\}}$ and λ_i is the deterministic intensity function of τ_i under $\mathbb{P}$, is a $(\mathbb{P}, \mathbb{H}^i)$ and a $(\mathbb{P}, \mathbb{G})$-martingale, where $\mathbb{G} =$
$hh^1 \vee \mathbb{H}^2$. (Recall that $\lambda_i(s)ds = \frac{\mathbb{P}(\tau_i \in ds)}{\mathbb{P}(\tau_i > s)}$). Manifestly, immersion hypothesis holds under $\mathbb{P}$ between $\mathbb{H}^1$ and $\mathbb{G}$. Let us set $d\mathbb{Q}\,|_{\mathcal{G}_t} = L_t\, d\mathbb{P}\,|_{\mathcal{G}_t}$, where

$$L_t = 1 + \int_0^t L_{u-} \kappa_u \, dM^1_u$$

for some $\mathbb{G}$-predictable process κ satisfying $\kappa > 1$ (WHY?). We set $\mathbb{F} = \mathbb{H}^1$ and $\mathbb{H} = \mathbb{H}^2$. Let

$$\begin{aligned}\widehat{M}_t^1 &= H_t^1 - \int_0^{t\wedge\tau_1} \widehat{\lambda}_1(u)\,du \\ \widetilde{M}_t^1 &= H_t^1 - \int_0^{t\wedge\tau_1} \lambda_1(u)(1+\kappa_u)\,du\end{aligned}$$

where $\widehat{\lambda}(u)du = \mathbb{Q}(\tau_1 \in du)/\mathbb{Q}(\tau_1 > u)$ is deterministic. It is easy to see that, under $\mathbb{Q}$, the process $\widehat{M}^1$ is a $(\mathbb{Q},\mathbb{H}^1)$-martingale and $\widetilde{M}^1$ is a $(\mathbb{Q},\mathbb{G})$ martingale. The process $\widehat{M}^1$ is not a $(\mathbb{Q},\mathbb{G})$-martingale (WHY?), hence, immersion does not hold under $\mathbb{Q}$.

Exercise 3.3.8 Compute $\mathbb{Q}(\tau_1 > t|\mathcal{H}_t^2)$. ◁

3.3.6 Ordered times

Assume that $\tau_i, i = 1,\dots,n$ are n random times. Let $\sigma_i, i = 1,\dots,n$ be the sequence of ordered random times and $\mathbb{G}^{(k)} = \mathbb{F}\vee\mathbb{H}^{(1)}\cdots\vee\mathbb{H}^{(k)}$ where $\mathbb{H}^{(i)} = (\mathcal{H}_t^{(i)} = \sigma(t\wedge\sigma_i), t\geq 0)$. The $\mathbb{G}^{(k)}$-intensity of σ_k is the positive $\mathbb{G}^{(k)}$-adapted process λ^k such that $(M_t^{(k)} := 1\!\!1_{\{\sigma_k\leq t\}} - \int_0^t \lambda_s^k ds, t\geq 0)$ is a $\mathbb{G}^{(k)}$-martingale. The $\mathbb{G}^{(k)}$-martingale $M^{(k)}$ is stopped at σ_k and the $\mathbb{G}^{(k)}$-intensity of σ_k satisfies $\lambda_t^k = 0$ on $\{t\geq\sigma_k\}$. The following lemma shows the $\mathbb{G}^{(k)}$-intensity of σ_k coincides with its $\mathbb{G}^{(n)}$-intensity.

Lemma 3.3.9 *For any k, a $\mathbb{G}^{(k)}$-martingale stopped at σ_k is a $\mathbb{G}^{(n)}$-martingale.*

PROOF: We prove that any $\mathbb{G}^{(1)}$-martingale stopped at σ_1 is a $\mathbb{G}^{(2)}$-martingale. The result will follow. Let X be a $\mathbb{G}^{(1)}$-martingale stopped at σ_1, i.e. $X_t = X_{t\wedge\sigma_1}$ for any t. For $s<t$,

$$\mathbb{E}[X_{t\wedge\sigma_1}|\mathcal{G}_s^{(2)}] = 1\!\!1_{\{\sigma_2\leq s\}}X_{\sigma_1} + 1\!\!1_{\{s<\sigma_2\}}\frac{\mathbb{E}[X_{t\wedge\sigma_1}1\!\!1_{\{s<\sigma_2\}}|\mathcal{G}_s^{(1)}]}{\mathbb{P}(s<\sigma_2|\mathcal{G}_s^{(1)})}$$

It remains to note that

$$\mathbb{E}[X_{t\wedge\sigma_1}1\!\!1_{\{s<\sigma_2\}}|\mathcal{G}_s^{(1)}] = 1\!\!1_{\{s<\sigma_1\}}\mathbb{E}[X_{t\wedge\sigma_1}|\mathcal{G}_s^{(1)}] + 1\!\!1_{\{\sigma_1\leq s\}}\mathbb{E}[X_{\sigma_1}1\!\!1_{\{s<\sigma_2\}}|\mathcal{G}_s^{(1)}]\,.$$

Since $\sigma_2 > s$ on $\{\sigma_1 > s\}$, we obtain $1\!\!1_{\{s<\sigma_1\}}\mathbb{P}(s<\sigma_2|\mathcal{G}_s^{(1)}) = 1\!\!1_{\{s<\sigma_1\}}$. The martingale property of X yields to

$$1\!\!1_{\{s<\sigma_1\}}\mathbb{E}[X_{t\wedge\sigma_1}|\mathcal{G}_s^{(1)}] = 1\!\!1_{\{s<\sigma_1\}}X_{s\wedge\sigma_1}$$

It is obvious that

$$1\!\!1_{\{\sigma_1\leq s\}}\mathbb{E}[X_{\sigma_1}1\!\!1_{\{s<\sigma_2\}}|\mathcal{G}_s^{(1)}] = 1\!\!1_{\{\sigma_1\leq s\}}X_{\sigma_1}\mathbb{P}(s<\sigma_2|\mathcal{G}_s^{(1)}).$$

The result follows. □

The following is a familiar result in the literature.

Proposition 3.3.10 *Assume that the $\mathbb{G}^{(k)}$-intensity λ^k of σ_k exists for all $k\in\Theta$. Then the intensity of the loss process $\sum_{k=1}^n 1\!\!1_{\sigma_k\leq t}$ is the sum of the intensities of σ_k, i.e.*

$$\lambda^L = \sum_{k=1}^n \lambda^k,\ a.s.. \tag{3.3.1}$$

PROOF: Since $(1\!\!1_{\{\sigma_k\leq t\}} - \int_0^t \lambda_s^k ds, t\geq 0)$ is a $\mathbb{G}^{(k)}$-martingale stopped at σ_k, it is a $\mathbb{G}^{(n)}$-martingale. We have by taking the sum that $(L_t - \int_0^t \sum_{k=1}^n \lambda_s^k ds, t\geq 0)$ is a $\mathbb{G}^{(n)}$-martingale. So $\lambda_t^L = \sum_{k=1}^n \lambda_t^k$ for all $t\geq 0$. □

Chapter 4

Bridges and utility maximization

The first applications of enlargement of filtration in Finance

4.1 The Brownian Bridge

Rather than studying ab initio the general problem of initial enlargement, we discuss an interesting example. Let us start with a BM $(B_t, t \geq 0)$ and its natural filtration $\mathbb{F}^B$. Define a new filtration as $\mathcal{F}_t^{(B_1)} = \mathcal{F}_t^B \vee \sigma(B_1)$. In this filtration, the process $(B_t, t \geq 0)$ is no longer a martingale. It is easy to be convinced of this by looking at the process $(\mathbb{E}(B_1|\mathcal{F}_t^{(B_1)}), t \leq 1)$: this process is identically equal to B_1, not to B_t, hence $(B_t, t \geq 0)$ is not a $\mathbb{G}$-martingale. However, $(B_t, t \geq 0)$ is a $\mathbb{F}^{(B_1)}$-semi-martingale, as follows from the next proposition 4.1.2.

Before giving this proposition, we recall some facts on Brownian bridge.

The **Brownian bridge** $(b_t, 0 \leq t \leq 1)$ is defined as the conditioned process $(B_t, t \leq 1 | B_1 = 0)$. Note that $B_t = (B_t - tB_1) + tB_1$ where, from the Gaussian property, the process $(B_t - tB_1, t \leq 1)$ and the random variable B_1 are independent. Hence $(b_t, 0 \leq t \leq 1) \stackrel{\text{law}}{=} (B_t - tB_1,\, 0 \leq t \leq 1)$. The Brownian bridge process is a Gaussian process, with zero mean and covariance function $s(1-t), s \leq t$. Moreover, it satisfies $b_0 = b_1 = 0$.

We can represent the Brownian bridge between 0 and y during the time interval $[0, 1]$ as

$$(B_t - tB_1 + ty;\ t \leq 1)\,.$$

More generally, the Brownian bridge between x and y during the time interval $[0, T]$ may be expressed as

$$\left(x + B_t - \frac{t}{T}B_T + \frac{t}{T}(y - x);\ t \leq T\right),$$

where $(B_t; t \leq T)$ is a standard BM starting from 0.

Exercise 4.1.1 a) Prove that the Riemann integral $\int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds$ is absolutely convergent.
b) Prove that, for $0 \leq s < t \leq 1$, $\mathbb{E}(B_t - B_s | B_1 - B_s) = \frac{t-s}{1-s}(B_1 - B_s)$ ◁

4.1.1 Decomposition of the BM in the enlarged filtration $\mathbb{F}^{(B_1)}$

Proposition 4.1.2 *Let* $\mathcal{F}_t^{(B_1)} = \cap_{\epsilon > 0} \mathcal{F}_{t+\epsilon} \vee \sigma(B_1)$. *The process*

$$\beta_t := B_t - \int_0^{t \wedge 1} \frac{B_1 - B_s}{1 - s} ds$$

is an $\mathbb{F}^{(B_1)}$-martingale, and an $\mathbb{F}^{(B_1)}$ Brownian motion. In other words,

$$B_t = \beta_t - \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} ds$$

is the decomposition of B as an $\mathbb{F}^{(B_1)}$-semi-martingale.

PROOF: Note that the definition of $\mathbb{F}^{(B_1)}$ is done to satisfy the right-continuity assumption. We shall note, as a short cut, $\mathcal{F}_t^{(B_1)} := \mathcal{F}_t \vee \sigma(B_1) = \mathcal{F}_t \vee \sigma(B_1 - B_t)$. Then, since $\mathcal{F}_s$ is independent of $(B_{s+h} - B_s, h \geq 0)$, one has, for $s < t$:

$$\mathbb{E}(B_t - B_s | \mathcal{F}_s^{(B_1)}) = \mathbb{E}(B_t - B_s | B_1 - B_s) = \frac{t-s}{1-s}(B_1 - B_s)\,.$$

For $s < t < 1$,

$$\begin{aligned}
\mathbb{E}(\int_s^t \frac{B_1 - B_u}{1-u} du | \mathcal{F}_s^{(B_1)}) &= \int_s^t \frac{1}{1-u} \mathbb{E}(B_1 - B_u | B_1 - B_s)\, du \\
&= \int_s^t \frac{1}{1-u} \left(B_1 - B_s - \mathbb{E}(B_u - B_s | B_1 - B_s)\right) du \\
&= \int_s^t \frac{1}{1-u} \left(B_1 - B_s - \frac{u-s}{1-s}(B_1 - B_s)\right) du \\
&= \frac{1}{1-s}(B_1 - B_s) \int_s^t du = \frac{t-s}{1-s}(B_1 - B_s)
\end{aligned}$$

It follows that

$$\mathbb{E}(\beta_t - \beta_s | \mathcal{F}_s^{(B_1)}) = 0$$

hence, β is an $\mathbb{F}^{(B_1)}$-martingale (and an $\mathbb{F}^{(B_1)}$-Brownian motion) (WHY?). □

It follows that if M is an $\mathbb{F}$-local martingale such that $\int_0^1 \frac{1}{\sqrt{1-s}} d|\langle M, B\rangle|_s$ is finite, then

$$\widehat{M}_t = M_t - \int_0^{t\wedge 1} \frac{B_1 - B_s}{1-s} d\langle M, B\rangle_s$$

is a $\mathbb{F}^{(B_1)}$-local martingale.

Comment 4.1.3 The singularity of $\frac{B_1 - B_t}{1-t}$ at $t = 1$, i.e., the fact that $\frac{B_1 - B_t}{1-t}$ is not square integrable between 0 and 1 prevents a Girsanov measure change transforming the $(\mathbb{P}, \mathbb{F}^{(B_1)})$ semi-martingale B into a $(\mathbb{Q}, \mathbb{F}^{(B_1)})$ martingale.

Comment 4.1.4 We obtain that the standard Brownian bridge b is a solution of the following stochastic equation (take care about the change of notation)

$$\left\{ \begin{array}{lcl} db_t &=& -\dfrac{b_t}{1-t} dt + dW_t\,;\; 0 \leq t < 1 \\ \\ b_0 &=& 0\,. \end{array} \right.$$

The solution of the above equation is $b_t = (1-t)\int_0^t \frac{1}{1-s} dW_s$ which is a Gaussian process with zero mean and covariance $s(1-t), s \leq t$.

Exercise 4.1.5 Using the notation of Proposition 4.1.2, prove that B_1 and β are independent. Check that the projection of β on $\mathbb{F}^B$ is equal to B.
`Hint:` The $\mathbb{F}^{(B_1)}$ BM β is independent of $\mathcal{F}_0^{(B_1)}$. ◁

Exercise 4.1.6 Consider the SDE

$$\begin{cases} dX_t &= -\dfrac{X_t}{1-t}dt + dW_t\,;\ 0 \le t < 1 \\ X_0 &= 0 \end{cases}$$

1. Prove that
$$X_t = (1-t)\int_0^t \frac{dW_s}{1-s}\,;\ 0 \le t < 1\,.$$
2. Prove that $(X_t, t \ge 0)$ is a Gaussian process. Compute its expectation and its covariance.
3. Prove that $\lim_{t\to 1} X_t = 0$.

◁

Exercise 4.1.7 (See Jeulin and Yor [76]) Let $X_t = \int_0^t \varphi_s dB_s$ where φ is predictable such that $\int_0^t \varphi_s^2 ds < \infty$. Prove that the following assertions are equivalent

1. X is an $\mathbb{F}^{(B_1)}$-semimartingale with decomposition
$$X_t = \int_0^t \varphi_s d\beta_s + \int_0^{t\wedge 1} \frac{B_1 - B_s}{t-s}\varphi_s ds$$
2. $\int_0^1 |\varphi_s|\frac{|B_1-B_s|}{1-s}ds < \infty$
3. $\int_0^1 \frac{|\varphi_s|}{\sqrt{1-s}}ds < \infty$

◁

4.2 Poisson Bridge

Let N be a Poisson process with constant intensity λ, $\mathcal{F}_t^N = \sigma(N_s, s \le t)$ its natural filtration and $T > 0$ a fixed time. The process $M_t = N_t - \lambda t$ is a martingale. Let $\mathcal{G}_t^* = \sigma(N_s, s \le t; N_T)$ be the natural filtration of N enlarged with the terminal value N_T of the process N.

Proposition 4.2.1 *Assume that $\lambda = 1$. The process*

$$\eta_t = M_t - \int_0^{t\wedge T} \frac{M_T - M_s}{T-s}ds,$$

is a $\mathbb{G}^*$-martingale with predictable bracket, for $t \le T$,

$$\Lambda_t = \int_0^t \frac{N_T - N_s}{T-s}ds\,.$$

PROOF: For $0 < s < t < T$,

$$\mathbb{E}(N_t - N_s|\mathcal{G}_s^*) = \mathbb{E}(N_t - N_s|N_T - N_s) = \frac{t-s}{T-s}(N_T - N_s)$$

where the last equality follows from the fact that, if X and Y are independent with Poisson laws with parameters μ and ν respectively, then

$$\mathbb{P}(X = k|X+Y = n) = \frac{n!}{k!(n-k)!}\alpha^k(1-\alpha)^{n-k}$$

where $\alpha = \dfrac{\mu}{\mu+\nu}$. Hence,

$$\begin{aligned}
\mathbb{E}\left(\int_s^t du \frac{N_T - N_u}{T-u} | \mathcal{G}_s^*\right) &= \int_s^t \frac{du}{T-u}\left(N_T - N_s - \mathbb{E}(N_u - N_s|\mathcal{G}_s^*)\right) \\
&= \int_s^t \frac{du}{T-u}\left(N_T - N_s - \frac{u-s}{T-s}(N_T - N_s)\right) \\
&= \int_s^t \frac{du}{T-s}(N_T - N_s) = \frac{t-s}{T-s}(N_T - N_s)\,.
\end{aligned}$$

Therefore,

$$\mathbb{E}(N_t - N_s - \int_s^t \frac{N_T - N_u}{T-u} du | \mathcal{G}_s^*) = \frac{t-s}{T-s}(N_T - N_s) - \frac{t-s}{T-s}(N_T - N_s) = 0$$

and the result follows.

□

Comment 4.2.2 Poisson bridges are studied in Jeulin and Yor [76]. This kind of enlargement of filtration is used for modelling insider trading in Elliott and Jeanblanc [44], Grorud and Pontier [57] and Kohatsu-Higa and Øksendal [85].

Exercise 4.2.3 Prove that, for any enlargement of filtration the compensated martingale M remains a semi-martingale.
Hint: M has bounded variation. ◁

Exercise 4.2.4 Prove that any $\mathbb{F}^N$-martingale is a $\mathbb{G}^*$-semimartingale. ◁

Exercise 4.2.5 Prove that

$$\eta_t = N_t - \int_0^{t\wedge T} \frac{N_T - N_s}{T-s} ds - (t-T)^+,$$

Prove that

$$\langle \eta \rangle_t = \int_0^{t\wedge T} \frac{N_T - N_s}{T-s} ds + (t-T)^+\,.$$

Therefore, $(\eta_t, t \leq T)$ is a compensated $\mathbb{G}^*$-Poisson process, time-changed by $\int_0^t \frac{N_T - N_s}{T-s} ds$, i.e., $\eta_t = \widetilde{M}(\int_0^t \frac{N_T - N_s}{T-s} ds)$ where $(\widetilde{M}(t), t \geq 0)$ is a compensated Poisson process. ◁

Exercise 4.2.6 A process X fulfills the **harness property** if

$$\mathbb{E}\left(\frac{X_t - X_s}{t-s}\Big| \mathcal{F}_{s_0],\,[T}\right) = \frac{X_T - X_{s_0}}{T - s_0}$$

for $s_0 \leq s < t \leq T$ where $\mathcal{F}_{s_0],\,[T} = \sigma(X_u, u \leq s_0, u \geq T)$. Prove that a process with the harness property satisfies

$$\mathbb{E}\left(X_t \Big| \mathcal{F}_{s],\,[T}\right) = \frac{T-t}{T-s} X_s + \frac{t-s}{T-s} X_T\,,$$

and conversely. Prove that, if X satisfies the harness property, then, for any fixed T,

$$M_t^T = X_t - \int_0^t du \frac{X_T - X_u}{T-u},\ t < T$$

is an $\mathcal{F}_{t],\,[T}$-martingale and conversely. See [3M] for more comments. ◁

4.3 Insider trading

In this section, we study a simple case of insider trading. We assume, that, in a BLack and Scholes model, an insider knows, at time 0 the value of the price at time 1. If the maturity of the market is 1, there are obviously arbitrage opportunity. We show how this insider can increase his wealth if the market terminates before date 1. We then study the same problem in a Poisson case.

4.3.1 Brownian Bridge

Let

$$dS_t = S_t(\mu dt + \sigma dB_t)$$

where μ and σ are constants, be the price of a risky asset. Assume that the riskless asset has an constant interest rate r.

The wealth of an agent holding ϑ^0 shares of the savings account and ϑ shares of the underlying risky process is $X_t = \vartheta^0_t e^{rt} + \vartheta_t S_t$. The self financing condition is that

$$dX_t = \vartheta^0_t de^{rt} + \vartheta_t dS_t = rX_t dt + \vartheta_t(dS_t - rS_t dt)$$

With the change of notation $\pi_t = \vartheta_t S_t / X_t$ (so that the wealth remains non negative) one has

$$dX_t = rX_t dt + \pi_t \sigma X_t(dW_t + \theta dt),\, X_0 = x$$

Here ϑ is the number of shares of the risky asset, and π the proportion of wealth invested in the risky asset. It follows that

$$\ln(X_T^{\pi,x}) = \ln x + \int_0^T (r - \frac{1}{2}\pi_s^2\sigma^2 + \theta\pi_s\sigma)ds + \int_0^T \sigma\pi_s dW_s$$

Then, assuming that the local martingale represented by the stochastic integral is in fact a martingale,

$$\mathbb{E}(\ln(X_T^{\pi,x})) = \ln x + \int_0^T \mathbb{E}\left(r - \frac{1}{2}\pi_s^2\sigma^2 + \theta\pi_s\sigma\right) ds$$

The portfolio which maximizes $\mathbb{E}(\ln(X_T^{\pi,x}))$ is $\pi_s = \frac{\theta}{\sigma}$ and

$$\sup \mathbb{E}(\ln(X_T^{\pi,x})) = \ln x + T\left(r + \frac{1}{2}\theta^2\right)$$

Note that, if the coefficients r, μ and σ are $\mathbb{F}$-adapted, the same computation leads to

$$\sup \mathbb{E}(\ln(X_T^{\pi,x})) = \ln x + \int_0^T \mathbb{E}\left(r_t + \frac{1}{2}\theta_t^2\right) dt$$

where $\theta_t = \frac{\mu_t - r_t}{\sigma_t}$.

We come back to the case of constant coefficients. We now enlarge the filtration with S_1 (or equivalently, with B_1 (WHY?)). In the enlarged filtration, setting, for $t < 1$, $\alpha_t = \frac{B_1 - B_t}{1-t}$, the dynamics of S are

$$dS_t = S_t((\mu + \sigma\alpha_t)dt + \sigma d\beta_t)\,,$$

where β is defined in Proposition 4.1.2 and the dynamics of the wealth are

$$dX_t = rX_t dt + \pi_t\sigma X_t(d\beta_t + \widetilde{\theta}_t dt),\, X_0 = x$$

with $\widetilde{\theta}_t = \frac{\mu-r}{\sigma} + \alpha_t = \frac{\mu-r}{\sigma} + \frac{B_1-B_s}{1-s}$. Assuming again that the stochastic integral which appears is a martingale, the portfolio which maximizes $\mathbb{E}(\ln(X_T^{\pi,x}))$ is $\pi_s = \frac{\widetilde{\theta}_s}{\sigma}$.
Then, for $T < 1$,

$$\begin{aligned}\ln(X_T^{\pi,x,*}) &= \ln x + \int_0^T (r + \frac{1}{2}\widetilde{\theta}_s^2)ds + \int_0^T \sigma\pi_s d\beta_s \\ \mathbb{E}(\ln(X_T^{\pi,x,*})) &= \ln x + \int_0^T (r + \frac{1}{2}(\theta^2 + \mathbb{E}(\alpha_s^2) + 2\theta\mathbb{E}(\alpha_s))ds = \ln x + (r + \frac{1}{2}\theta^2)T + \frac{1}{2}\int_0^T \mathbb{E}(\alpha_s^2)ds\end{aligned}$$

where we have used the fact that $\mathbb{E}(\alpha_t) = 0$ (if the coefficients r, μ and σ are $\mathbb{F}$ adapted, α is orthogonal to $\mathcal{F}_t$, hence $\mathbb{E}(\alpha_t\theta_t) = 0$). Let

$$\begin{aligned}V^{\mathbb{F}}(x) &= \max \mathbb{E}(\ln(X_T^{\pi,x}))\,;\ \pi \text{ is } \mathbb{F} \text{ adapted} \\ V^{\mathbb{G}}(x) &= \max \mathbb{E}(\ln(X_T^{\pi,x}))\,;\ \pi \text{ is } \mathbb{G} \text{ adapted}\end{aligned}$$

Then $V^{\mathbb{G}}(x) = V^{\mathbb{F}}(x) + \frac{1}{2}\mathbb{E}\int_0^T \alpha_s^2 ds = V^{\mathbb{F}}(x) - \frac{1}{2}\ln(1-T)$.

If $T = 1$, the value function is infinite: there is an arbitrage opportunity and there does not exist an e.m.m. such that the discounted price process $(e^{-rt}S_t, t \le 1)$ is a $\mathbb{G}$-martingale. However, for any $\epsilon \in]0,1]$, there exists a uniformly integrable $\mathbb{G}$-martingale L defined as

$$dL_t = \frac{\mu - r + \sigma\zeta_t}{\sigma} L_t d\beta_t,\ t \le 1-\epsilon, \quad L_0 = 1\,,$$

such that, setting $d\mathbb{Q}|_{\mathcal{G}_t} = L_t d\mathbb{P}|_{\mathcal{G}_t}$, the process $(e^{-rt}S_t, t \le 1-\epsilon)$ is a $(\mathbb{Q}, \mathbb{G})$-martingale.

This is the main point in the theory of insider trading where the knowledge of the terminal value of the underlying asset creates an arbitrage opportunity, which is effective at time 1.

It is important to mention, that in both cases, the wealth of the investor is $X_t e^{-rt} = x + \int_0^t \pi_s d(S_s e^{-rs})$. The insider has a larger class of portfolio, and in order to give a meaning to the stochastic integral for processes π which are not adapted with respect to the semi-martingale S, one has to give the decomposition of this semi-martingale in the larger filtration.

Exercise 4.3.1 Prove carefully that there does not exist any emm in the enlarged filtration. Make precise the arbitrage opportunity. $\triangleleft$

4.3.2 Poisson Bridge

We suppose that the interest rate is null and that the risky asset has dynamics

$$dS_t = S_{t-}\left(\mu dt + \sigma dW_t + \phi dM_t\right)$$

where M is the compensated martingale of a standard Poisson process. Let $(X_t, t \ge 0)$ be the wealth of an un-informed agent whose portfolio is described by (π_t), the proportion of wealth invested in the asset S at time t. Then

$$dX_t = \pi_t X_{t-}(\mu dt + \sigma dW_t + \phi dM_t) \tag{4.3.1}$$

Then,

$$X_t = x\exp\left(\int_0^t \pi_s(\mu - \phi\lambda)ds + \int_0^t \sigma\pi_s dW_s + \frac{1}{2}\int_0^t \sigma^2\pi_s^2 ds + \int_0^t \ln(1 + \pi_s\phi)dN_s\right)$$

Assuming that the stochastic integrals with respect to W and M are martingales,

$$\mathbb{E}[\ln(X_T)] = \ln(x) + \int_0^T \mathbb{E}(\mu\pi_s - \frac{1}{2}\sigma^2\pi_s^2 + \lambda(\ln(1 + \phi\pi_s) - \phi\pi_s)ds\,.$$

Our aim is to solve

$$V(x) = \sup_\pi \mathbb{E}\left(\ln(X_T^{x,\pi})\right)$$

We can then maximize the quantity under the integral sign for each s and ω.
The maximum attainable wealth for the uninformed agent is obtained using the constant strategy $\tilde{\pi}$ for which

$$\tilde{\pi}\mu + \lambda[\ln(1+\tilde{\pi}\phi) - \tilde{\pi}\phi] - \frac{1}{2}\tilde{\pi}^2\sigma^2 = \sup_\pi[\pi\mu + \lambda[\ln(1+\pi\phi) - \pi\phi] - \frac{1}{2}\pi^2\sigma^2] .$$

Hence

$$\tilde{\pi} = \frac{1}{2\sigma^2\phi}\left(\mu\phi - \phi^2\lambda - \sigma^2 \pm \sqrt{(\mu\phi - \phi^2\lambda - \sigma^2)^2 + 4\sigma^2\phi\mu}\right) .$$

The quantity under the square root is $(\mu\phi - \phi^2\lambda + \sigma^2)^2 + 4\sigma^2\phi^2\lambda$ and is non-negative.
The sign to be used depends on the sign of quantities related to the parameters. The optimal $\widetilde{\pi}$ is the only one such that $1 + \phi\widetilde{\pi} > 0$. Solving the equation (4.3.1), it can be proved that the optimal wealth is $\widetilde{X}_t = x(\widetilde{L}_t)^{-1}$ where $d\widetilde{L}_t = \widetilde{L}_{t-}(-\sigma\widetilde{\pi}dW_t + (\frac{1}{1+\phi\widetilde{\pi}} - 1)dM_t)$ is a Radon Nikodym density of an equivalent martingale measure. In this incomplete market, we thus obtain the utility equivalent martingale measure defined by Davis [32] and duality approach (See Kramkov and Schachermayer).

We assume now that the informed agent knows N_T from time 0. Therefore, his wealth evolves according to the dynamics

$$dX_t^* = \pi_t X_{t-}^*[(\mu + \phi(\Lambda_t - \lambda)]dt + \sigma dW_t + \phi dM_t^*]$$

where Λ is given in Proposition 4.2.1. Exactly the same computations as above can be carried out. In fact these only require changing μ to $(\mu + \phi(\Lambda_t - \lambda))$ and the intensity of the jumps from λ to Λ_t. The optimal portfolio π^* is now such that $\mu - \lambda\phi + \phi\Lambda_s[\frac{1}{1+\pi^*\phi}] - \pi^*\sigma^2 = 0$ and is given by

$$\pi_s^* = \frac{1}{2\sigma^2\phi}\left(\mu\phi - \phi^2\lambda - \sigma^2 \pm \sqrt{(\mu\phi - \phi^2\lambda + \sigma^2)^2 + 4\sigma^2\phi^2\Lambda_s}\right) ,$$

The optimal wealth is $X_t^* = x(L_t^*)^{-1}$ where

$$dL_t^* = L_{t-}^*(-\sigma\pi_s^* dW_t + (\frac{1}{1+\phi\pi_s^*} - 1)dM_t^*) .$$

Whereas the optimal portfolio of the uninformed agent is constant, the optimal portfolio of the informed agent is time-varying and has a jump as soon as a jump occurs for the prices.
The informed agent must maximize at each (s, ω) the quantity

$$\pi\mu + \Lambda_s(\omega)\ln(1+\pi\phi) - \lambda\pi\phi - \frac{1}{2}\pi^2\sigma^2 .$$

Consequently,

$$\sup_\pi \pi\mu + \Lambda_s \ln(1+\pi\phi) - \lambda\pi\phi - \frac{1}{2}\pi^2\sigma^2 \geq \tilde{\pi}\mu + \Lambda_s\ln(1+\tilde{\pi}\phi) - \lambda\tilde{\pi}\phi - \frac{1}{2}\tilde{\pi}^2\sigma^2$$

Now, $\mathbb{E}[\Lambda_s] = \lambda$, so

$$\begin{aligned}\sup_\pi \mathbb{E}(\ln X_T^*) &= \ln x + \sup_\pi \int_0^T \mathbb{E}(\pi\mu + \Lambda_s\ln(1+\pi\phi) - \lambda\pi\phi - \frac{1}{2}\pi^2\sigma^2)ds \\ &\geq \ln x + \int_0^T \tilde{\pi}(\mu + \lambda\ln(1+\tilde{\pi}\phi) - \lambda\tilde{\pi}\phi - \frac{1}{2}\tilde{\pi}^2\sigma^2)ds = \mathbb{E}(\ln \widetilde{X}_T)\end{aligned}$$

Therefore, the maximum expected wealth for the informed agent is greater than that of the uninformed agent. This is obvious because the informed agent can use any strategy available to the uninformed agent.

Exercise 4.3.2 Solve the same problem for power utility function. ◁

4.4 Drift information in a Progressive Enlargement in a Brownian setting

We assume in this part that W is a Brownian motion with natural filtration $\mathbb{F}$ and $\mathbb{G}$ is a filtration larger than $\mathbb{F}$ and that there there exists an integrable $\mathbb{G}$-adapted process $\mu^{\mathbb{G}}$ such that $dW_t = dW_t^{\mathbb{G}} + \mu_t^{\mathbb{G}} dt$ where $W^{\mathbb{G}}$ is a $\mathbb{G}$-BM. We study a financial market where a risky asset with price S (an $\mathbb{F}$-adapted positive process) and a riskless asset $S^0 \equiv 1$ are traded is arbitrage free. More precisely, we assume w.l.g. that S is a $(\mathbb{P}, \mathbb{F})$ (local) martingale, $dS_t = S_t \sigma_t dW_t$.

Let X be the wealth process associated with a $\mathbb{G}$ predictable strategy

$$dX_t = \widehat{\pi}_t dS_t = \widehat{\pi}_t S_t dW_t = \pi_t X_t dW_t = \pi_t X_t (dW_t^{\mathbb{G}} + \mu_t^{\mathbb{G}} dt)$$

so that

$$X_t = x \exp\left(\int_0^t \pi_s dW_s^{\mathbb{G}} - \frac{1}{2} \int_0^t \pi_s^2 ds + \int_0^t \pi_s \mu_s^{\mathbb{G}} ds \right)$$

It is then easy to see that the optimal π is $\pi = \mu^{\mathbb{G}}$ and that

$$\ln X_t^* = \ln x + \int_0^t \pi_s^* dW_s^{\mathbb{G}} + \frac{1}{2} \int_0^t (\mu_s^{\mathbb{G}})^2 ds$$

so that

$$\sup_{\pi \in \mathbb{F}} \mathbb{E}(\ln X_T) = \ln x < \sup_{\pi \in \mathbb{G}} \mathbb{E}(\ln X_T) = \ln x + \mathbb{E}\left(\frac{1}{2} \int_0^t (\mu_s^{\mathbb{G}})^2 ds \right)$$

which leads to a finite utility if

$$\mathbb{E}\left(\int_0^t (\mu_s^{\mathbb{G}})^2 ds \right) < \infty$$

Note that, if $L_t := \mathcal{E}(-\mu^{\mathbb{G}} W^{\mathbb{G}})_t$ is a martingale, NFLVR holds, and if L is a local martingales, one say that the No arbitrages of the first kind holds (there exists a positive local martingale L such that SL is a $\mathbb{P}$ local martingale).

Chapter 5

Initial Enlargement

In this chapter, we study initial enlargement, where the enlarged filtration is $\mathcal{F}_t^{(L)} = \mathcal{F}_t \vee \sigma(L)$ for a random variable L. The goal is to give conditions such that $\mathbb{F}$-martingales remain $\mathbb{F}^{(L)}$-semi-martingales and, in that case, to give the $\mathbb{F}^{(L)}$-semi-martingale decomposition of the $\mathbb{F}$-martingales.

More precisely, in order to satisfy the usual hypotheses, redefine

$$\mathcal{F}_t^{(L)} = \cap_{\epsilon>0} \{\mathcal{F}_{t+\epsilon} \vee \sigma(L)\} \ .$$

We denote $\mathcal{P}(\mathbb{F})$ the predictable σ-algebra (see Subsection 1.1.3).

5.1 General Facts

Proposition 5.1.1 *One has*
(i) Every $\mathcal{F}_t^{(L)}$-measurable r.v. Y_t is of the form $Y_t(\omega) = y_t(\omega, L(\omega))$ for some $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$-measurable random variable $(y_t(\omega, u), t \geq 0)$.
(ii) Every $\mathbb{F}^{(L)}$-predictable process Y is of the form $Y_t(\omega) = y_t(\omega, L(\omega))$ where $(t, \omega, u) \mapsto y_t(\omega, u)$ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable function.

PROOF: The proof of part (i) is based on the fact that $\mathcal{F}_t^{(L)}$-measurable random variables are generated by random variables of the form $X_t(\omega) = x_t(\omega) f\big(L(\omega)\big)$, with $x_t \in \mathcal{F}_t$ and f bounded Borel function on $\mathbb{R}$.

(ii) It suffices to notice that processes of the form $X_t := x_t f(L)$, $t \geq 0$, where x is $\mathbb{F}$-predictable and f is a bounded Borel function on $\mathbb{R}$, generate the $\mathcal{F}^{(L)}$-predictable σ-field. □

We shall now simply write $y_t(L)$ for $y_t(\omega, L(\omega))$.

5.2 An absolute continuity result

We recall that there exists a family of regular conditional distributions $P_t(\omega, dx)$ such that $P_t(\cdot, A)$ is a version of $\mathbb{P}(L \in A | \mathcal{F}_t)$ and for any ω, $P_t(\omega, \cdot)$ is a probability on $\mathbb{R}$.

5.2.1 Jacod's criterion

In what follows, for $y(u)$ a family of martingales and X a martingale, we shall write $\langle y(L), X \rangle$ for $\langle y(u), X \rangle|_{u=L}$.

Proposition 5.2.1 (Jacod's Criterion.) *Suppose that, for each $t < T$, $P_t(\omega, dx) \ll \nu(dx)$ where ν is the law of L and T is a fixed horizon $T \leq \infty$. Then, every $\mathbb{F}$-semi-martingale $(X_t, t < T)$ is also an $\mathbb{F}^{(L)}$-semi-martingale.*
Moreover, if $P_t(\omega, dx) = p_t(\omega, x)\nu(dx)$, the process $p(L)$ does not vanish on $[0, T[$ and if X is an $\mathbb{F}$-martingale, the process

$$\widetilde{X}_t = X_t - \int_0^t \frac{d\langle p.(L), X\rangle_s}{p_{s-}(L)}\,, t < T$$

is an $\mathbb{F}^{(L)}$-martingale. In other words, the decomposition of the $\mathbb{F}^{(L)}$-semi-martingale X is

$$X_t = \widetilde{X}_t + \int_0^t \frac{d\langle p.(L), X\rangle_s}{p_{s-}(L)}\,.$$

Proof: In a first step, we show that, for any θ, the process $p(\theta) = (p_t(\theta), t \geq 0)$ is an $\mathbb{F}$-martingale. One has to show that, for a bounded r.v. $\zeta_s \in \mathcal{F}_s$ and $s < t$

$$\mathbb{E}(p_t(\theta)\zeta_s) = \mathbb{E}(p_s(\theta)\zeta_s)$$

This follows from

$$\mathbb{E}(\mathbb{E}(\mathbb{1}_{\tau>\theta}|\mathcal{F}_t)\zeta_s) = \mathbb{E}(\mathbb{E}(\mathbb{1}_{\tau>\theta}|\mathcal{F}_s)\zeta_s)\,.$$

In a second step, we assume that $\mathbb{F}$-martingales are continuous (condition **(C)**), and that X and p are square integrable. In that case, $\langle p.(L), X\rangle$ exists. Let F_s be a bounded $\mathcal{F}_s$-measurable random variable and $h : \mathbb{R}^+ \to \mathbb{R}$, be a bounded Borel function. Then the variable $F_s h(L)$ is $\mathcal{F}_s^{(L)}$-measurable and if a decomposition of the form $X_t = \widetilde{X}_t + \int_0^t dK_u(L)$ holds, the martingale property of $\widetilde{X}$ should imply that $\mathbb{E}\left(F_s h(L)\left(\widetilde{X}_t - \widetilde{X}_s\right)\right) = 0$, hence

$$\mathbb{E}\left(F_s h(L)\left(X_t - X_s\right)\right) = \mathbb{E}\left(F_s h(L) \int_s^t dK_u(L)\right).$$

We can write:

$$\begin{aligned}
\mathbb{E}\left(F_s h(L)\left(X_t - X_s\right)\right) &= \mathbb{E}\left(F_s\left(X_t - X_s\right)\int_{-\infty}^{\infty} h(\theta)p_t(\theta)\nu(d\theta)\right) \\
&= \int_{\mathbb{R}} h(\theta)\mathbb{E}\left(F_s\left(X_t p_t(\theta) - X_s p_s(\theta)\right)\right)\nu(d\theta) \\
&= \int_{\mathbb{R}} h(\theta)\mathbb{E}\left(F_s \int_s^t d\left\langle X, p(\theta)\right\rangle_v\right)\nu(d\theta)
\end{aligned}$$

where the first equality comes from a conditioning w.r.t. $\mathcal{F}_t$, the second from the martingale property of $p(\theta)$, and the third from the fact that both X and $p(\theta)$ are square-integrable $\mathbb{F}$-martingales. Moreover:

$$\begin{aligned}
\mathbb{E}\left(F_s h(L)\int_s^t dK_v(L)\right) &= \mathbb{E}\left(F_s \int_{\mathbb{R}} h(\theta)\int_s^t dK_v(\theta)p_t(\theta)\nu(d\theta)\right) \\
&= \int_{\mathbb{R}} h(\theta)\mathbb{E}\left(F_s \int_s^t p_v(\theta)dK_v(\theta)\right)\nu(d\theta)
\end{aligned}$$

where the first equality comes from the definition of p, and the second from the martingale property of $p(\theta)$. By equalization of these two quantities, we obtain that it is necessary to have $dK_u(\theta) = d\left\langle X, p(\theta)\right\rangle_u / p_u(\theta)$. □

For the general case, we refer the reader to Jacod.

Remark 5.2.2 Of course, if for each $t \leq T$, $P_t(\omega, dx) \ll \nu(dx)$ where ν is the law of L, every $\mathbb{F}$-semi-martingale $(X_t, t \leq T)$ is also an $\mathbb{F}^{(L)}$-semi-martingale. In many cases, the hypothesis is not satisfied for T (see the Brownian bridge case).

Definition 5.2.3 *We shall say that L satisfies AC-hypothesis if*

$$\mathbb{P}(L \in dx|\mathcal{F}_t) = P_t(\omega, dx) = p_t(\omega, x)\nu(dx)$$

The stability of AC-hypothesis under a change of probability is rather obvious.

Corollary 5.2.4 *Let Z be a random variable taking only a countable number of values. Then every $\mathbb{F}$ semimartingale is a $\mathbb{F}^{(Z)}$ semimartingale.*

PROOF: If we note

$$\eta\,(dx) = \sum_{k=1}^{\infty} \mathbb{P}\,(Z = x_k)\,\delta_{x_k}\,(dx)\,,$$

where $\delta_{x_k}(dx)$ is the Dirac measure at x_k, the law of Z, then $P_t(\omega, dx)$ is absolutely continuous with respect to η with Radon-Nikodym density:

$$\sum_{k=1}^{\infty} \frac{\mathbb{P}\,(Z = x_k|\mathcal{F}_t)}{\mathbb{P}\,(Z = x_k)} \mathbb{1}_{x=x_k}.$$

Now the result follows from Jacod's theorem.□

Exercise 5.2.5 Assume that $\mathbb{F}$ is a Brownian filtration. Then, $\mathbb{E}(\int_0^t \frac{d\langle p_\cdot(L), X\rangle_s}{p_{s-}(L)}|\mathcal{F}_t)$ is an $\mathbb{F}$-martingale. ◁

5.2.2 Regularity Conditions

One of the major difficulties is to prove the existence of a universal càdlàg martingale version of the family of densities. Fortunately, results of Jacod [64] or Stricker and Yor [107] help us to solve this technical problem. See also [5] for a detailed discussion. We emphazise that these results are the most important part of enlargement of filtration theory.

Jacod ([64], Lemme 1.8 and 1.10) establishes the existence of a universal càdlàg version of the density process in the following sense: there exists a non negative function $p_t(\omega, \theta)$ càdlàg in t, optional w.r.t. the filtration $\widehat{\mathbb{F}}$ on $\widehat{\Omega} = \Omega \times \mathbb{R}^+$, generated by $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$, such that

- for any θ, $p_\cdot(\theta)$ is an $\mathbb{F}$-martingale; moreover, denoting $\zeta^\theta = \inf\{t : p_{t-}(\theta) = 0\} \wedge T$, then $p_\cdot(\theta) > 0$, and $p_-(\theta) > 0$ on $[0, \zeta^\theta)$, and $p_\cdot(\theta) = 0$ on $[\zeta^\theta, T)$. Furthermore, $\zeta^L = T$, $\mathbb{P}$-a.s.
- For any bounded family $(Y_t(\omega, \theta), t \geq 0)$ measurable w.r.t. $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$, the $\mathbb{F}$-predictable projection of the process $Y_t(\omega, L(\omega))$ is the process $Y_t^{(p)} = \int p_{t-}(\theta) Y_t(\theta)\nu(d\theta)$.
- If $(\omega, t, \theta) \rightarrow) Y_t(\omega, \theta)$ is non negative and $\mathcal{O}(\mathbb{F})) \otimes \mathcal{B}$ measurable, the optional projection of the process $Y(L)$ is $\int Y_t(\theta) p_t(\theta)\nu(d\theta)$.
- Let m be a local $\mathbb{F}$-martingale. There exists predictable increasing process A and a $\widehat{\mathbb{F}}$-predictable function k such that

  $$\langle p(\theta), m\rangle_t = \int_0^t k_s(\theta) p_{s-}(\theta) dA_s\,.$$

  If m is locally square integrable, one can chose $A = \langle m\rangle$.

Exercise 5.2.6 Prove that if there exists a probability $\mathbb{Q}^*$ equivalent to $\mathbb{P}$ such that, under $\mathbb{Q}^*$, the r.v. L is independent of $\mathcal{F}_\infty$, then every $(\mathbb{P}, \mathbb{F})$-semi-martingale X is also an $(\mathbb{P}, \mathbb{F}^{(L)})$-semi-martingale. See Chapter 8. ◁

5.3 Yor's Method

We follow here Yor [115] (see also [114]). We assume that $\mathbb{F}$ is a Brownian filtration. For a bounded Borel function f, let $(\lambda_t(f), t \geq 0)$ be the continuous version of the martingale $(\mathbb{E}(f(L)|\mathcal{F}_t), t \geq 0)$. There exists a predictable kernel $\lambda_t(dx)$ such that

$$\lambda_t(f) = \int_{\mathbb{R}} \lambda_t(dx) f(x) \,.$$

From the predictable representation property applied to the martingale $\mathbb{E}(f(L)|\mathcal{F}_t)$, there exists a predictable process $\widehat{\lambda}(f)$ such that

$$\lambda_t(f) = \mathbb{E}(f(L)) + \int_0^t \widehat{\lambda}_s(f) dB_s \,.$$

Proposition 5.3.1 *We assume that there exists a predictable kernel $\widehat{\lambda}_t(dx)$ such that*

$$dt \text{ a.s.}, \quad \widehat{\lambda}_t(f) = \int_{\mathbb{R}} \widehat{\lambda}_t(dx) f(x) \,.$$

Assume furthermore that $dt \times d\mathbb{P}$ a.s. the measure $\widehat{\lambda}_t(dx)$ is absolutely continuous with respect to $\lambda_t(dx)$:

$$\widehat{\lambda}_t(dx) = \rho(t,x) \lambda_t(dx) \,.$$

Then, if X is an $\mathbb{F}$-martingale, there exists a $\mathbb{F}^{(L)}$-martingale $\widehat{X}$ such that

$$X_t = \widehat{X}_t + \int_0^t \rho(s,L) d\langle X, B\rangle_s \,.$$

Sketch of the proof: Let X be an $\mathbb{F}$-martingale, f a given bounded Borel function and $F_t = \mathbb{E}(f(L)|\mathcal{F}_t)$. From the hypothesis

$$F_t = \mathbb{E}(f(L)) + \int_0^t \widehat{\lambda}_s(f) dB_s \,.$$

Then, for $A_s \in \mathcal{F}_s$, $s < t$:

$$\begin{aligned} \mathbb{E}(\mathbb{1}_{A_s} f(L)(X_t - X_s)) &= \mathbb{E}(\mathbb{1}_{A_s}(F_t X_t - F_s X_s)) = \mathbb{E}(\mathbb{1}_{A_s}(\langle F, X\rangle_t - \langle F, X\rangle_s)) \\ &= \mathbb{E}\left(\mathbb{1}_{A_s} \int_s^t d\langle X, B\rangle_u \, \widehat{\lambda}_u(f)\right) \\ &= \mathbb{E}\left(\mathbb{1}_{A_s} \int_s^t d\langle X, B\rangle_u \int_{\mathbb{R}} \lambda_u(dx) f(x) \rho(u,x)\right) . \end{aligned}$$

Therefore, $V_t = \int_0^t \rho(u,L)\, d\langle X, B\rangle_u$ satisfies

$$\mathbb{E}(\mathbb{1}_{A_s} f(L)(X_t - X_s)) = \mathbb{E}(\mathbb{1}_{A_s} f(L)(V_t - V_s)) \,.$$

It follows that, for any $G_s \in \mathcal{F}_s^{(L)}$,

$$\mathbb{E}(\mathbb{1}_{G_s}(X_t - X_s)) = \mathbb{E}(\mathbb{1}_{G_s}(V_t - V_s)) \,,$$

hence, $(X_t - V_t, t \geq 0)$ is an $\mathbb{F}^{(L)}$-martingale. □

Let us write the result of Proposition 5.3.1 in terms of Jacod's criterion. If $\lambda_t(dx) = p_t(x)\nu(dx)$, then

$$\lambda_t(f) = \int p_t(x) f(x) \nu(dx) \,.$$

Hence,

$$d\langle\lambda.(f),B\rangle_t = \widehat{\lambda}_t(f)dt = \int dx f(x)\, d_t\langle p.(x),B\rangle_t$$

and

$$\widehat{\lambda}_t(dx) = d_t\langle p.(x),B\rangle_t = \frac{d_t\langle p.(x),B\rangle_t}{p_t(x)}p_t(x)dx$$

therefore,

$$\widehat{\lambda}_t(dx)dt = \frac{d_t\langle p.(x),B\rangle_t}{p_t(x)}\lambda_t(dx)\,.$$

In the case where $\lambda_t(dx) = \Phi(t,x)dx$, with $\Phi > 0$, it is possible to find ψ such that

$$\Phi(t,x) = \Phi(0,x)\exp\left(\int_0^t \psi(s,x)dB_s - \frac{1}{2}\int_0^t \psi^2(s,x)ds\right)$$

and it follows that $\widehat{\lambda}_t(dx) = \psi(t,x)\lambda_t(dx)$. Then, if X is an $\mathbb{F}$-martingale of the form $X_t = x + \int_0^t x_s dB_s$, the process $(X_t - \int_0^t ds\, x_s\, \psi(s,L), t \geq 0)$ is an $\mathbb{F}^{(L)}$-martingale.

5.4 Examples

We now give some examples taken from Mansuy & Yor [93] in a Brownian set-up for which we use the preceding. Here, B is a standard Brownian motion.
See Jeulin [73] and Mansuy & Yor [93] for more examples. ✓Add Amendinger exemple

5.4.1 Enlargement with B_1.

We compare the results obtained in Subsection 4.1 and the method presented in Subsection 5.3. Let $L = B_1$. From the Markov property

$$\mathbb{E}(g(B_1)|\mathcal{F}_t) = \mathbb{E}(g(B_1 - B_t + B_t)|\mathcal{F}_t) = F_g(B_t, 1-t)$$

where $F_g(y,1-t) = \int g(x)P(1-t;y,x)dx$ and $P(s;y,x) = \frac{1}{\sqrt{2\pi s}}\exp\left(-\frac{(x-y)^2}{2s}\right)$. It follows that $\lambda_t(dx) = \frac{1}{\sqrt{2\pi(1-t)}}\exp\left(-\frac{(x-B_t)^2}{2(1-t)}\right)dx$. Then

$$\lambda_t(dx) = p_t(x)\mathbb{P}(B_1 \in dx) = p_t(x)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$

with

$$p_t(x) = \frac{1}{\sqrt{(1-t)}}\exp\left(-\frac{(x-B_t)^2}{2(1-t)} + \frac{x^2}{2}\right)\,.$$

From Itô's formula,

$$d_t p_t(x) = p_t(x)\frac{x-B_t}{1-t}dB_t\,.$$

(This can be considered as a partial check of the martingale property of $(p_t(x), t \geq 0)$.) It follows that $d\langle p(x),B\rangle_t = p_t(x)\frac{x-B_t}{1-t}dt$, hence

$$B_t = \widetilde{B}_t + \int_0^t \frac{B_1 - B_s}{1-s}ds\,.$$

Note that, in the notation of Proposition 5.3.1, one has

$$\widehat{\lambda}_t(dx) = \frac{x-B_t}{1-t}\frac{1}{\sqrt{2\pi(1-t)}}\exp\left(-\frac{(x-B_t)^2}{2(1-t)}\right)dx\,.$$

5.4.2 Enlargement with $M^B = \sup_{s\le 1} B_s$.

From Exercise 1.5.1,

$$\mathbb{E}(f(M^B)|\mathcal{F}_t) = F(1-t, B_t, M_t^B)$$

where $M_t^B = \sup_{s\le t} B_s$ with

$$F(s,a,b) = \sqrt{\frac{2}{\pi s}}\left(f(b)\int_0^{b-a} e^{-u^2/(2s)}du + \int_b^\infty f(u)e^{-(u-a)^2/(2s)}du\right)$$

and, denoting by δ_y the Dirac measure at y,

$$\lambda_t(dy) = \sqrt{\frac{2}{\pi(1-t)}}\left\{\delta_y(M_t^B)\int_0^{M_t^B-B_t}\exp\left(-\frac{u^2}{2(1-t)}\right)du + 1\!\!1_{\{y>M_t^B\}}\exp\left(-\frac{(y-B_t)^2}{2(1-t)}\right)dy\right\}.$$

Hence, by applying Itô's formula

$$\widehat{\lambda}_t(dy) = \sqrt{\frac{2}{\pi(1-t)}}\left\{\delta_y(M_t^B)\exp\left(-\frac{(M_t^B-B_t)^2}{2(1-t)}\right) + 1\!\!1_{\{y>M_t^B\}}\frac{y-B_t}{1-t}\exp\left(-\frac{(y-B_t)^2}{2(1-t)}\right)\right\}.$$

It follows that

$$\rho(t,x) = 1\!\!1_{\{x>M_t^B\}}\frac{x-B_t}{1-t} + 1\!\!1_{\{M_t^B=x\}}\frac{1}{\sqrt{1-t}}\frac{\Phi'}{\Phi}\left(\frac{x-B_t}{\sqrt{1-t}}\right)$$

with $\Phi(x) = \sqrt{\frac{2}{\pi}}\int_0^x e^{-\frac{u^2}{2}}du$.

5.4.3 Enlargement with $\int_0^\infty e^{2B_s^{(-\mu)}}ds$

Consider $A_t^{(\mu)} := \int_0^t e^{2B_s^{(\mu)}}ds$ where $B_t^{(\mu)} = B_t + \mu t$, μ being a positive constant. Matsumoto and Yor [94] have established that $A_\infty^{(-\mu)} = A_t^{(-\mu)} + e^{2B_t^{(-\mu)}}\widetilde{A}_\infty^{(-\mu)}$ where $\widetilde{A}_\infty^{(-\mu)}$ is independent of $\mathcal{F}_t$, with the same law as $A_\infty^{(-\mu)}$. The law of $A_\infty^{(-\mu)}$ is proved to be the law of $1/(2\gamma_\mu)$, γ_μ being a Gamma random variable with parameter μ, i.e., admits the survival probability of $\Upsilon(x) = \frac{1}{\Gamma(\mu)}\int_0^{1/(2x)} y^{\mu-1}e^{-y}dy$, where Γ is the Gamma function. Then, one obtains

$$G_t(\theta) = P(A_\infty^{(-\mu)} > \theta|\mathcal{F}_t) = \Upsilon\Big(\frac{\theta - A_t^{(-\mu)}}{e^{2B_t^{(-\mu)}}}\Big)1\!\!1_{\theta > A_t^{(-\mu)}} + 1\!\!1_{\theta\le A_t^{(-\mu)}}.$$

This gives a family of martingale survival processes G with gamma structure. It follows that, on $\{\theta > A_t^{(-\mu)}\}$

$$dG_t(\theta) = \frac{1}{2^{\mu-1}\Gamma(\mu)}e^{-\frac{1}{2}Z_t(\theta)}(Z_t(\theta))^\mu dB_t$$

where $Z_t(\theta) = \frac{e^{2B_t^{(-\mu)}}}{\theta - A_t^{(-\mu)}}$ (to have light notation, we do not specify that this process Z depends on μ). One can check that $G_t(\cdot)$ is differentiable w.r.t. θ, so that $G_t(\theta) = \int_\theta^\infty g_t(u)du$, where

$$g_t(u) = 1_{u>A_t^{(-\mu)}}\frac{1}{2^\mu\Gamma(\mu)}(Z_t(u))^{\mu+1}e^{-\frac{1}{2}Z_t(u) - 2B_t^{(-\mu)}}.$$

✓It remains to compute dg_t

5.4.4 Enlargement with $L := \int_0^\infty f(s)dB_s$

Let B be a Brownian motion with natural filtration $\mathbb{F}$ and $L = \int_0^\infty f(s)dB_s$ where f is a deterministic function such that $\int_0^\infty f^2(s)ds < \infty$. The above method applies step by step: it is easy to compute $\lambda_t(dx)$, since conditionally on $\mathcal{F}_t$, L is Gaussian, with mean $m_t = \int_0^t f(s)\,dB_s$, and variance $\sigma^2(t) = \int_t^\infty f^2(s)\,ds$. Since $\mathbb{P}(L \le x|\mathcal{F}_t) = \Phi(\frac{x-m_t}{\sigma(t)})$, where Φ is the cumulative distribution function of a standard gaussian law, the absolute continuity requirement is satisfied with:

$$p_t(x)\nu(dx) = \frac{1}{\sigma(t)}\varphi(\frac{x-m_t}{\sigma(t)})dx,$$

where φ is the density of a standard Gaussian law, and ν the law of Z (a centered Gaussian law with variance $\sigma^2(0)$). Note that, from Itô's calculus,

$$dp_t(x) = p_t(x)\frac{x-m_t}{\sigma^2(t)}dm_t$$

But here, we have to impose an extra integrability condition. For example, if we assume that

$$\int_0^t \frac{|f(s)|}{\sigma(s)}ds < \infty,$$

then B is a $\mathbb{F}^{(L)}$-semimartingale with canonical decomposition:

$$B_t = \widetilde{B}_t + \int_0^t ds\frac{f(s)}{\sigma^2(s)}\left(\int_s^\infty f(u)\,dB_u\right),$$

As a particular case, we may take $L = B_{t_0}$, for some fixed t_0 and we recover the results for the Brownian bridge.

5.5 A Particular Change of Probability

We assume now that P_t, the conditional law of L given $\mathcal{F}_t$, is **equivalent** to ν for any $t > 0$, so that the hypotheses of Proposition 5.2.1 hold and that p does not vanishes (on the support of ν). See Chapter 8 for more comments.

Lemma 5.5.1 *The process $1/p_t(L), t \ge 0$ is an $\mathbb{F}^{(L)}$-martingale. Let, for any t, $d\mathbb{Q} = 1/p_t(L)d\mathbb{P}$ on $\mathcal{F}_t^{(L)}$. Under $\mathbb{Q}$, the r.v. L is independent of $\mathcal{F}_\infty$. Moreover,*

$$\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t},\ \mathbb{Q}|_{\sigma(L)} = \mathbb{P}|_{\sigma(L)}$$

PROOF: Let $Z_t(L) = 1/p_t(L)$. In a first step, we prove that Z is an $\mathbb{F}^{(L)}$-martingale. Indeed, $\mathbb{E}(Z_t(L)|\mathcal{F}_s^{(L)}) = Z_s(L)$ if (and only if) $\mathbb{E}(Z_t(L)h(L)A_s) = \mathbb{E}(Z_s(L)h(L)A_s)$ for any (bounded) Borel function h and any $\mathcal{F}_s$-measurable (bounded) random variable A_s. From definition of p, one has

$$\begin{aligned}\mathbb{E}(Z_t(L)h(L)A_s) &= \mathbb{E}(\int_{\mathbb{R}} Z_t(x)h(x)p_t(x)\nu(dx)A_s) = \mathbb{E}(\int_{\mathbb{R}} h(x)\nu(dx)A_s)\\ &= \int_{\mathbb{R}} h(x)\nu(dx)\mathbb{E}(A_s) = \mathbb{E}(A_s)\,\mathbb{E}(h(L))\end{aligned}$$

The particular case $t = s$ leads to $\mathbb{E}(Z_s(L)h(L)A_s) = \mathbb{E}(h(L))\,\mathbb{E}(A_s)$, hence $\mathbb{E}(Z_s(L)h(L)A_s) = \mathbb{E}(Z_t(L)h(L)A_s)$. Note that, since $p_0(x) = 1$, one has $\mathbb{E}(1/p_t(L)|\mathcal{F}_0^{(L)}) = 1/p_0(L) = 1$.
Now, we prove the required independence. From the above,

$$\mathbb{E}_{\mathbb{Q}}(h(L)A_s) = \mathbb{E}_{\mathbb{P}}(Z_s(L)h(L)A_s) = \mathbb{E}_{\mathbb{P}}(h(L))\,\mathbb{E}_{\mathbb{P}}(A_s)$$

For $h = 1$ (resp. $A_t = 1$), one obtains $\mathbb{E}_{\mathbb{Q}}(A_s) = \mathbb{E}_{\mathbb{P}}(A_s)$ (resp. $\mathbb{E}_{\mathbb{Q}}(h(L)) = \mathbb{E}_{\mathbb{P}}(h(L))$) and we are done. □

This fact appears in Song [105] and plays an important rôle in Grorud and Pontier [58].

Lemma 5.5.2 *Let m be a $(\mathbb{P}, \mathbb{F})$ martingale. The process m^L defined by $m_t^L = m_t/p_t(L)$ is a $(\mathbb{P}, \mathbb{F}^{(L)})$-martingale and satisfies $\mathbb{E}(m_t^L|\mathcal{F}_t) = m_t$.*

PROOF: To establish the martingale property, it suffices to check that for $s < t$ and $A \in \mathcal{F}_s^{(L)}$, one has $\mathbb{E}_{\mathbb{P}}(m_t^L \mathbb{1}_A) = \mathbb{E}_{\mathbb{P}}(m_s^L \mathbb{1}_A)$, which is equivalent to $\mathbb{E}_{\mathbb{Q}}(m_t \mathbb{1}_A) = \mathbb{E}_{\mathbb{Q}}(m_s \mathbb{1}_A)$. The last equality follows from the fact that the $(\mathbb{F}, \mathbb{P})$ martingale m is also a $(\mathbb{F}, \mathbb{Q})$ martingale (indeed $\mathbb{P}$ and $\mathbb{Q}$ coincide on $\mathbb{F}$), hence a $(\mathbb{F}^{(L)}, \mathbb{Q})$ martingale (by independence of L and $\mathbb{F}$ under $\mathbb{Q}$. Bayes criteria shows that m^L) is a $(\mathbb{P}, \mathbb{F}^{(L)})$-martingale. Noting that $\mathbb{E}(1/p_t(L)|\mathcal{F}_t) = 1$ (take $A_s = 1$ and $h = 1$ in the preceeding proof), the equality

$$\mathbb{E}(m_t^L|\mathcal{F}_t) = m_t \mathbb{E}(1/p_t(L)|\mathcal{F}_t) = m_t$$

ends the proof. □

Of course, the reverse holds true: if there exists a probability equivalent to $\mathbb{P}$ such that, under $\mathbb{Q}$, the r.v. L is independent to $\mathcal{F}_\infty$, then $(\mathbb{P}, \mathbb{F})$-martingales are $(\mathbb{P}, \mathbb{F}^{(L)})$-semi martingales (WHY?).

Exercise 5.5.3 Prove that $(Y_t(L), t \geq 0)$ is a $(\mathbb{P}, \mathbb{F}^{(L)})$-martingale if and only if $Y_t(x)p_t(x)$ is a family of $\mathbb{F}$-martingales. ◁

Exercise 5.5.4 Let $\mathbb{F}$ be a Brownian filtration. Prove that, if X is a square integrable $(\mathbb{P}, \mathbb{F}^{(L)})$-martingale, then, there exists a function h and a process ψ such that

$$X_t = h(L) + \int_0^t \psi_s(L) dB_s$$

◁

5.5.1 Equivalent Martingale Measures

We consider a financial market, where some assets, with prices S are traded, as well as a riskless asset S^0. The dynamics of S under the so-called historical probability are assumed to be semi-martingales (WHY?). We denote by $\mathbb{F}$ the filtration generated by prices and by $\mathcal{Q}^{\mathbb{F}}$ the set of probability measures, equivalent to $\mathbb{P}$ on $\mathbb{F}$ such that the discounted prices $\widetilde{S}$ are $\mathbb{F}$-(local) martingales. We now enlarge the filtration and consider prices as $\mathbb{F}^{(L)}$ semi-martingales (assuming that it is the case). We denote by $\mathcal{Q}^{(L)}$ the set of probability measures, equivalent to $\mathbb{P}$ on $\mathbb{F}^{(L)}$ such that the discounted prices $\widetilde{S}$ are $\mathbb{F}^{(L)}$-(local) martingales.

We assume that there exists a probability $\mathbb{P}^*$ equivalent to $\mathbb{P}$ such that, under $\mathbb{P}^*$, the r.v. L is independent to $\mathcal{F}_\infty$. Then, $\mathcal{Q}^{(L)}$ is equal to the set of probability measures, equivalent to $\mathbb{P}^*$ on $\mathbb{F}^{(L)}$ such that the discounted prices $\widetilde{S}$ are $\mathbb{F}^{(L)}$-(local) martingales.

✓To be completed

5.5.2 Enlargement with S_∞

Proposition 5.5.5 *Let N be a local martingale such that its supremum process S is continuous (this is the case if N is in the class $\mathcal{C}_0$). Let f be a locally bounded Borel function and define $F(x) = \int_0^x dy f(y)$. Then, $X_t := F(S_t) - f(S_t)(S_t - N_t)$ is a local martingale and:*

$$F(S_t) - f(S_t)(S_t - N_t) = \int_0^t f(S_s) dN_s + F(S_0), \tag{5.5.1}$$

PROOF: In the case of a Brownian motion (i.e., $N = B$), this was done in Exercise 1.5.2. In the case of continuous martingales, if F is C^2,

$$\begin{aligned} F(S_t) - f(S_t)(S_t - N_t) &= F(S_t) - \int_0^t f(S_s)\,dS_s + \int_0^t f(S_s)\,dN_s \\ &\quad + \int_0^t (S_s - N_s) f'(S_s) dS_s \end{aligned}$$

The last integral is null, because dS is carried by $\{S - N = 0\}$ and $\int_0^t f(S_s)\,dS_s = F(S_t) - F(S_0)$. For the general case, we refer the reader to [99]. □

Let us introduce $\mathcal{F}_t^{(S_\infty)} = \mathcal{F}_t \vee \sigma(S_\infty))$. Since $g = \sup\{t : N_t = S_\infty\}$, the random variable g is an $\mathbb{F}^{(S_\infty)}$-stopping time. Consequently $\mathcal{F}_t^g \subset \mathcal{F}_t^{(S_\infty)}$.

Proposition 5.5.6 *For any Borel bounded or positive function f, we have:*

$$\mathbb{E}(f(S_\infty)|\mathcal{F}_t) = f(S_t)\left(1 - \frac{N_t}{S_t}\right) + \int_0^{N_t/S_t} dx f\left(\frac{N_t}{x}\right)$$

PROOF: In the following, U is a random variable, which follows the standard uniform law and which is independent of $\mathcal{F}_t$.

$$\begin{aligned} \mathbb{E}(f(S_\infty)|\mathcal{F}_t) &= \mathbb{E}\left(f\left(S_t \vee S^t\right)|\mathcal{F}_t\right) \\ &= \mathbb{E}\left(f(S_t)\,\mathbb{1}_{\{S_t \geq S^t\}}|\mathcal{F}_t\right) + \mathbb{E}\left(f\left(S^t\right)\mathbb{1}_{\{S_t < S^t\}}|\mathcal{F}_t\right) \\ &= f(S_t)\,\mathbb{P}\left(S_t \geq S^t|\mathcal{F}_t\right) + \mathbb{E}\left(f\left(S^t\right)\mathbb{1}_{\{S_t < S^t\}}|\mathcal{F}_t\right) \\ &= f(S_t)\,\mathbb{P}\left(U \leq \frac{N_t}{S_t}|\mathcal{F}_t\right) + \mathbb{E}\left(f\left(\frac{N_t}{U}\right)\mathbb{1}_{\{U < \frac{N_t}{S_t}\}}|\mathcal{F}_t\right) \\ &= f(S_t)\left(1 - \frac{N_t}{S_t}\right) + \int_0^{N_t/S_t} dx f\left(\frac{N_t}{x}\right). \end{aligned}$$

□

We now show that $\mathbb{E}(f(S_\infty)|\mathcal{F}_t)$ is of the form 5.5.1. A straightforward change of variable in the last integral also gives:

$$\begin{aligned} \mathbb{E}(f(S_\infty)|\mathcal{F}_t) &= f(S_t)\left(1 - \frac{N_t}{S_t}\right) + N_t \int_{S_t}^\infty dy \frac{f(y)}{y^2} \\ &= f(S_t)\left(1 - \frac{N_t}{S_t}\right) + N_t \int_{S_t}^\infty dy \frac{f(y)}{y^2} \\ &= S_t \int_{S_t}^\infty dy \frac{f(y)}{y^2} - (S_t - N_t)\left(\int_{S_t}^\infty dy \frac{f(y)}{y^2} - \frac{f(S_t)}{S_t}\right). \end{aligned}$$

Hence,

$$\mathbb{E}(f(S_\infty)|\mathcal{F}_t) = H(1) + H(S_t) - h(S_t)(S_t - N_t),$$

with

$$H(x) = x \int_x^\infty dy \frac{f(y)}{y^2},$$

and

$$h(x) = h_f(x) \equiv \int_x^\infty dy \frac{f(y)}{y^2} - \frac{f(x)}{x} = \int_x^\infty \frac{dy}{y^2}(f(y) - f(x)).$$

Moreover, from the Azéma-Yor type formula 5.5.1, we have the following representation of $\mathbb{E}(f(S_\infty)|\mathcal{F}_t)$ as a stochastic integral:

$$\mathbb{E}(f(S_\infty)|\mathcal{F}_t) = \mathbb{E}(f(S_\infty)) + \int_0^t h(S_s)\,dN_s.$$

Moreover, there exist two families of random measures $(\lambda_t(dx))_{t\geq 0}$ and $\left(\dot{\lambda}_t(dx)\right)_{t\geq 0}$, with

$$\begin{aligned}
\lambda_t(dx) &= \left(1-\frac{N_t}{S_t}\right)\delta_{S_t}(dx) + N_t \mathbb{1}_{\{x>S_t\}}\frac{dx}{x^2} \\
\dot{\lambda}_t(dx) &= -\frac{1}{S_t}\delta_{S_t}(dx) + \mathbb{1}_{\{x>S_t\}}\frac{dx}{x^2},
\end{aligned}$$

such that

$$\begin{aligned}
\mathbb{E}(f(S_\infty)|\mathcal{F}_t) = \lambda_t(f) &= \int \lambda_t(dx) f(x) \\
\dot{\lambda}_t(f) &= \int \dot{\lambda}_t(dx) f(x).
\end{aligned}$$

Finally, we notice that there is an absolute continuity relationship between $\lambda_t(dx)$ and $\dot{\lambda}_t(dx)$; more precisely,

$$\dot{\lambda}_t(dx) = \lambda_t(dx)\rho(x,t),$$

with

$$\rho(x,t) = \frac{-1}{S_t - N_t}\mathbb{1}_{\{S_t=x\}} + \frac{1}{N_t}\mathbb{1}_{\{S_t<x\}}.$$

Theorem 5.5.7 *Let N be a local martingale in the class $\mathcal{C}_0$ (recall $N_0=1$). Then, any $\mathbb{F}$ martingale X is a $\mathbb{F}^{(S_\infty)}$-emimartingale with canonical decomposition:*

$$X_t = \widetilde{X}_t + \int_0^t \mathbb{1}_{\{g>s\}}\frac{d\langle X,N\rangle_s}{N_{s-}} - \int_0^t \mathbb{1}_{\{g\leq s\}}\frac{d\langle X,N\rangle_s}{S_\infty - N_{s-}},$$

where $\widetilde{X}$ is a $\mathbb{F}^{(S_\infty)}$-local martingale.

Proof: We can first assume that X is in $\mathbb{H}^1$; the general case follows by localization. Let K_s be an $\mathcal{F}_s$ measurable set, and take $t>s$. Then, for any bounded test function f, $\lambda_t(f)$ is a bounded martingale, hence in BMO, and we have:

$$\begin{aligned}
\mathbb{E}(\mathbb{1}_{K_s}f(S_\infty)(X_t-X_s)) &= \mathbb{E}(\mathbb{1}_{K_s}(\lambda_t(f)X_t - \lambda_s(f)X_s)) \\
&= \mathbb{E}(\mathbb{1}_{K_s}(\langle\lambda(f),X\rangle_t - \langle\lambda(f),X\rangle_s)) \\
&= \mathbb{E}\left(\mathbb{1}_{K_s}\left(\int_s^t \dot{\lambda}_u(f)\,d\langle X,N\rangle_u\right)\right) \\
&= \mathbb{E}\left(\mathbb{1}_{K_s}\left(\int_s^t\int \lambda_u(dx)\rho(x,u)f(x)\,d\langle X,N\rangle_u\right)\right) \\
&= \mathbb{E}\left(\mathbb{1}_{K_s}\left(\int_s^t d\langle X,N\rangle_u \rho(S_\infty,u)\right)\right).
\end{aligned}$$

But we also have:

$$\rho(S_\infty,t) = \frac{-1}{S_t-N_t}\mathbb{1}_{\{S_t=S_\infty\}} + \frac{1}{N_t}\mathbb{1}_{\{S_t<S_\infty\}}.$$

It now suffices to use the fact that S is constant after g and g is the first time when $S_\infty = S_t$, or in other words:

$$\mathbb{1}_{\{S_\infty>S_t\}} = \mathbb{1}_{\{g>t\}}, \text{ and } \mathbb{1}_{\{S_\infty=S_t\}} = \mathbb{1}_{\{g\leq t\}}.$$

Chapter 6

Filtering

In this chapter, our goal is to show how one can apply the idea of change of probability framework to a filtering problem (due to Kallianpur and Striebel [78]), to obtain the Kallianpur-Striebel formula for the conditional density (see also Meyer [96]). Our results are established in a very simple way, in a general filtering model, when the signal is a random variable, and contain, in the simple case, the results of Filipovic et al. [50]. We end the section with the examples of the traditional Gaussian filtering problem and of disorder.

6.1 Change of probability measure

One starts with the elementary model where, on the filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$, an $\mathcal{A}$-measurable random variable X is independent from the reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and its law admits a density probability g_0, so that

$$\mathbb{P}(X > \theta | \mathcal{F}_t) = \mathbb{P}(X > \theta) = \int_\theta^\infty g_0(u) du \,.$$

We denote by $\mathbb{F}^{(X)} = \mathbb{F} \vee \sigma(X)$ the filtration generated by $\mathbb{F}$ and X.
Let $(\beta_t(u), t \in I\!\!R_+)$ be a family of positive $(\mathbb{P}, \mathbb{F})$-martingales such that $\beta_0(u) = 1$ for all $u \in I\!\!R$. Note that, due to the assumed independence of X and $\mathbb{F}$, the process $(\beta_t(X), t \geq 0)$ is an $\mathbb{F}^{(X)}$-martingale and one can define a probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_t^{(X)})$, by $d\mathbb{Q} = \beta_t(X) d\mathbb{P}$. Since $\mathbb{F}$ is a subfiltration of $\mathbb{F}^{(X)}$, the positive $\mathbb{F}$-martingale

$$m_t^\beta := \mathbb{E}(\beta_t(X) | \mathcal{F}_t) = \int_{-\infty}^\infty \beta_t(u) g_0(u) du$$

is the Radon-Nikodým density of the measure $\mathbb{Q}$, restricted to $\mathcal{F}_t$ with respect to $\mathbb{P}$ (note that $m_0^\beta = 1$). Moreover, the $\mathbb{Q}$-conditional density of X with respect to $\mathcal{F}_t$ can be computed, from the Bayes' formula

$$\mathbb{Q}(X \in B | \mathcal{F}_t) = \frac{1}{\mathbb{E}(\beta_t(X) | \mathcal{F}_t)} \mathbb{E}(\mathbb{1}_B(X) \beta_t(X) | \mathcal{F}_t) = \frac{1}{m_t^\beta} \int_B \beta_t(u) g_0(u) du$$

where we have used, in the last equality the independence between X and $\mathbb{F}$, under $\mathbb{P}$. Let us summarize this simple but important result:

Proposition 6.1.1 *If X is a r.v. with probability density g_0, independent from $\mathbb{F}$ under $\mathbb{P}$, and if $\mathbb{Q}$ is a probability measure, equivalent to $\mathbb{P}$ on $\mathbb{F} \vee \sigma(X)$ with Radon-Nikodým density $\beta_t(X), t \geq 0$,* *then the* $(\mathbb{Q}, \mathbb{F})$ *density process of* X *is*

$$g_t^{\mathbb{Q}}(u)du := \mathbb{Q}(X \in du|\mathcal{F}_t) = \frac{1}{m_t^\beta}\,\beta_t(u)g_0(u)du \tag{6.1.1}$$

where m^β *is the normalizing factor* $m_t^\beta = \int_{-\infty}^{\infty} \beta_t(u)g_0(u)du$. *In particular*

$$\mathbb{Q}(\tau \in du) = \mathbb{P}(\tau \in du) = g_0(u)du\,.$$

The right-hand side of (6.1.1) can be understood as the ratio of $\beta_t(u)g_0(u)$ (the change of probability times the $\mathbb{P}$ probability density) and a normalizing coefficient m_t^β. One can say that $(\beta_t(u)g_0(u), t \geq 0)$ is the un-normalized density, obtained by a linear transformation from the initial density. The normalization factor m_t^β introduces a nonlinear dependence of $g_t^{\mathbb{Q}}(u)$ with respect to the initial density.

Remark 6.1.2 We present here some important remarks.
(1) If, for any t, $m_t^\beta = 1$, then the probability measures $\mathbb{P}$ and $\mathbb{Q}$ coincide on $\mathbb{F}$.
(2) Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be the usual right-continuous and complete filtration in the default framework (i.e. when $X = \tau$ is a non negative r.v.) generated by $\mathcal{F}_t \vee \sigma(\tau \wedge t)$. Similar calculation may be made with respect to $\mathcal{G}_t$. The only difference is that the conditional distribution of τ is a Dirac mass on the set $\{t \geq \tau\}$. On the set $\{\tau > t\}$, and under $\mathbb{Q}$, the distribution of τ admits a density given by:

$$\mathbb{Q}(\tau \in du|\mathcal{G}_t) = \beta_t(u)g_0(u)\frac{1}{\int_t^\infty \beta_t(\theta)g_0(\theta)d\theta}du\,.$$

(3) This methodology can be easily extended to a multivariate setting: one starts with an elementary model, where the $\tau_i, i = 1, \ldots, d$ are independent from $\mathbb{F}$, with joint density $g(u_1, \ldots, u_d)$. With a family of non-negative martingales $\beta(\theta_1, \ldots, \theta_d)$, the associated change of probability provides a multidimensional density process.

6.2 Filtering theory

The change of probability approach presented in the previous Section 6.1 is based on the idea that, in order to present modesl with a conditional density, one can restrict our attention to the simple case where the random variable is independent from the filtration and use a change of probability. The same idea is the building block of filtering theory as we present now.

Let W be a Brownian motion on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and X be a random variable independent of W, with probability density g_0. We denote by

$$dY_t = a(t, Y_t, X)dt + b(t, Y_t)dW_t \tag{6.2.1}$$

the observation process, where a and b are smooth enough to have a solution and where b does not vanish. The goal is to compute the conditional density of X with respect to the filtration $\mathbb{F}^Y$. The way we shall solve the problem is to construct a probability $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that, under $\mathbb{Q}$, the signal X and the observation $\mathbb{F}^Y$ are independent, and to compute the density of X under $\mathbb{P}$ by means of the change of probability approach of the previous section. It is known in nonlinear filtering theory as the Kallianpur-Striebel methodology [78], a way to linearize the problem.
Note that, from the independence assumption between X and W, we see that W is a $\mathbb{F}^{(X)} = \mathbb{F}^W \vee \sigma(X)$-martingale under $\mathbb{P}$.

6.2.1 Simple case

We start with the simple case where the dynamics of the observation is

$$dY_t = a(t, X)dt + dW_t\,.$$

We assume that a is smooth enough so that the solution of

$$d\beta_t(X) = -\beta_t(X)a(t,X)dW_t,\ \beta_0(X) = 1$$

is a $(\mathbb{P}, \mathbb{F}^{(X)})$-martingale, and we define a probability measure $\mathbb{Q}$ on $\mathcal{F}_t^{(X)}$ by $d\mathbb{Q} = \beta_t(X)d\mathbb{P}$. Then, by Girsanov's theorem, the process Y is a $(\mathbb{Q}, \mathbb{F}^{(X)})$-Brownian motion, hence is independent from $\mathbb{F}_0^{(X)} = \sigma(X)$, under $\mathbb{Q}$. Then, we apply our change of probability methodology, writing

$$d\mathbb{P} = \frac{1}{\beta_t(X)} d\mathbb{Q} =: \ell_t(X)d\mathbb{Q}$$

with

$$d\ell_t(X) = \ell_t(X)a(t,X)dY_t,\ \ell_0(X) = 1\,;$$

in other words, $\ell_t(u) = \frac{1}{\beta_t(u)} = \exp\left(\int_0^t a(s,u)dY_s - \frac{1}{2}\int_0^t a^2(s,u)ds\right)$. From Proposition 6.1.1, we obtain that the density of X under $\mathbb{P}$, with respect to $\mathbb{F}^Y$, is $g_t(u)$, given by

$$\mathbb{P}(X \in du|\mathcal{F}_t^Y) = g_t(u)du = \frac{1}{m_t^\ell} g_0(u)\ell_t(u)du$$

where $m_t^\ell = \mathbb{E}_\mathbb{Q}(\ell_t(X)|\mathcal{F}_t^Y) = \int_{-\infty}^\infty \ell_t(u)g_0(u)du$. Using the fact that

$$dm_t^\ell = \left(\int_{-\infty}^\infty \ell_t(u)a(t,u)g_0(u)du\right) dY_t = m_t^\ell \left(\int_{-\infty}^\infty g_t(u)a(t,u)du\right) dY_t$$

and setting

$$\widehat{a}_t := \mathbb{E}(a(t,X)|\mathcal{F}_t^Y) = \int_{-\infty}^\infty g_t(u)a(t,u)du\,,$$

Girsanov's theorem implies that the process B given by

$$dB_t = dY_t - \widehat{a}_t dt = dW_t + (a(t,X) - \widehat{a}_t)\,dt$$

is a $(\mathbb{P}, \mathbb{F}^Y)$ Brownian motion (it is the innovation process). From Itô's calculus, it is easy to show that the density process satisfies the nonlinear filtering equation

$$\begin{aligned} dg_t(u) &= g_t(u)\left(a(t,u) - \frac{1}{m_t^\ell}\int_{-\infty}^\infty dy\, g_0(y)a(t,y)\ell_t(y)\right) dB_t \\ &= g_t(u)\,(a(t,u) - \widehat{a}_t)\,dB_t\,. \end{aligned} \tag{6.2.2}$$

Remark 6.2.1 Observe that conversely, given a solution $g_t(u)$ of (6.2.2), and the process μ solution of $d\mu_t = \mu_t \widehat{a}_t dY_t$, then $h_t(u) = \mu_t g_t(u)$ is solution of the linear equation $dh_t(u) = h_t(u)a(t,u)dY_t$.

6.2.2 General case

Using the same ideas, we now solve the filtering problem in the case where the observation follows (6.2.1). Let $\beta(X)$ be the $\mathbb{F}^{(X)}$ local martingale, solution of

$$d\beta_t(X) = \beta_t(X)\sigma_t(X)dW_t,\ \beta_0(X) = 1$$

with $\sigma_t(X) = -\frac{a(t,Y_t,X)}{b(t,Y_t)}$. We assume that a and b are smooth enough so that β is a martingale. Let $\mathbb{Q}$ be defined on $\mathcal{F}_t^{(X)}$ by $d\mathbb{Q} = \beta_t(X)d\mathbb{P}$.
From Girsanov's theorem, the process $\widehat{W}$ defined as

$$d\widehat{W}_t = dW_t - \sigma_t(X)dt = \frac{1}{b(t,Y_t)}dY_t$$

is a $(\mathbb{Q}, \mathbb{G}^X)$-Brownian motion, hence $\widehat{W}$ is independent from $\mathcal{G}_0^X = \sigma(X)$. Being $\mathbb{F}^Y$-adapted, the process $\widehat{W}$ is a $(\mathbb{Q}, \mathbb{F}^Y)$-Brownian motion, X is independent from $\mathbb{F}^Y$ under $\mathbb{Q}$, and, as mentioned in Proposition 6.1.1, admits, under $\mathbb{Q}$, the probability density g_0.
We now assume that the natural filtrations of Y and $\widehat{W}$ are the same. To do so, note that it is obvious that $\mathbb{F}^{\widehat{W}} \subseteq \mathbb{F}^Y$. If the SDE $dY_t = b(t, Y_t)d\widehat{W}_t$ has a strong solution (e.g., if b is Lipschitz, with linear growth) then $\mathbb{F}^Y \subseteq \mathbb{F}^{\widehat{W}}$ and the equality between the two filtrations holds.
Then, we apply our change of probability methodology, with $\mathbb{F}^Y$ as the reference filtration, writing $d\mathbb{P} = \ell_t(X)d\mathbb{Q}$ with $d\ell_t(X) = -\ell_t(X)\sigma_t(X)d\widehat{W}_t$ (which follows from $\ell_t(X) = \frac{1}{\beta_t(X)}$) and we get that the density of X under $\mathbb{P}$, with respect to $\mathbb{F}^Y$ is $g_t(u)$ given by

$$g_t(u) = \frac{1}{m_t^\ell} g_0(u)\ell_t(u)$$

with dynamics

$$\begin{aligned} dg_t(u) &= -g_t(u)\left(\sigma_t(u) - \frac{1}{m_t^\ell}\int_{-\infty}^{\infty} dy\, g_0(y)\sigma_t(y)\ell_t(y)\right) dB_t \\ &= g_t(u)\left(\frac{a(t,Y_t,u)}{b(t,Y_t)} - \frac{1}{b(t,Y_t)}\int_{-\infty}^{\infty} dy\, g_t(y)a(t,Y_t,y)\right) dB_t \\ &= g_t(u)\left(\frac{a(t,Y_t,u)}{b(t,Y_t)} - \frac{\widehat{a}_t}{b(t,Y_t)}\right) dB_t\,. \end{aligned} \tag{6.2.3}$$

Here B is a $(\mathbb{P}, \mathbb{F}^Y)$ Brownian motion (the innovation process) given by

$$dB_t = dW_t + \left(\frac{a(t,Y_t,X)}{b(t,Y_t)} - \frac{\widehat{a}_t}{b(t,Y_t)}\right) dt\,,$$

where $\widehat{a}_t = \mathbb{E}(a(t,Y_t,X)|\mathcal{F}_t^Y)$.

Proposition 6.2.2 *If the signal X has probability density $g_0(u)$ and is independent from the Brownian motion W, and if the observation process Y follows*

$$dY_t = a(t,Y_t,X)dt + b(t,Y_t)dW_t\,,$$

then, the conditional density of X given $\mathcal{F}_t^Y$ is

$$\mathbb{P}(X \in du|\mathcal{F}_t^Y) = g_t(u)du = \frac{1}{m_t^\ell} g_0(u)\ell_t(u)du \tag{6.2.4}$$

where $\ell_t(u) = \exp\left(\int_0^t \frac{a(s,Y_s,u)}{b^2(s,Y_s)}dY_s - \frac{1}{2}\int_0^t \frac{a^2(s,Y_s,u)}{b^2(s,Y_s)}ds\right)$, $m_t^\ell = \int_{-\infty}^{\infty} \ell_t(u)g_0(u)du$, and its dynamics is given in (6.2.3).

6.2.3 General case

Assume now that X has a non trivial conditional law w.r.t. the Brownian motion driving the observation process. We assume that

$$\mathbb{P}(X > u|\mathcal{F}_t^W) = \int_u^\infty p_t(v)dv$$

and that the observation is

$$dY_t = a(t,Y_t,X)dt + b(t,Y_t)dW_t$$

Then, the process

$$B_t := W_t + \int_0^t \frac{d\langle p.(\theta), W\rangle_s|_{\theta=X}}{p_s(X)}$$

is a $\mathbb{F}^W \vee \sigma(X)$ Brownian motion, independent of X. It follows that

$$dY_t = \left(a(t, Y_t, X)dt - b(t, Y_t)\frac{d\langle p.(\theta), W\rangle_t|_{\theta=X}}{p_t(X)} \right) + b(t, Y_t)dB_t$$

and we can apply the previous results.

6.2.4 Gaussian filter

We apply our results to the well known case of Gaussian filter. Let W be a Brownian motion, X a random variable (the signal) with density probability g_0 a Gaussian law with mean m_0 and variance γ_0, independent of the Brownian motion W and let Y (the observation) be the solution of

$$dY_t = (a_0(t, Y_t) + a_1(t, Y_t)X)dt + b(t, Y_t)dW_t,$$

Then, from the previous results, the density process $g_t(u)$ is of the form

$$\frac{1}{m_t^\ell} \exp\left(\int_0^t \frac{a_0(s, Y_s) + a_1(s, Y_s)u}{b^2(s, Y_s)} \, dY_t - \frac{1}{2} \int_0^t \left(\frac{a_0(s, Y_s) + a_1(s, Y_s)u}{b(s, Y_s)} \right)^2 ds \right) g_0(u)$$

The logarithm of $g_t(u)$ is a quadratic form in u with stochastic coefficient, so that $g_t(u)$ is a Gaussian density, with mean m_t and variance γ_t (as proved already by Liptser and Shiryaev [91]). A tedious computation, purely algebraic, shows that

$$\gamma_t = \frac{\gamma_0}{1 + \gamma_0 \int_0^t \frac{a_1^2(s, Y_s)}{b^2(s, Y_s)} ds}, \quad m_t = m_0 + \int_0^t \gamma_s \frac{a_1(s, Y_s)}{b(s, Y_s)} \, dB_s$$

with $dB_t = dW_t + \frac{a_1(t, Y_t)}{b(t, Y_t)}(X - \mathbb{E}(X|\mathcal{F}_t^Y))dt$.

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In the case where the coefficients of the process Y are deterministic functions of time, i.e.,

$$dY_t = (a_0(t) + a_1(t)X)dt + b(t)dW_t$$

the variance $\gamma(t)$ is deterministic and the mean m is an $\mathbb{F}^Y$-Gaussian martingale

$$\gamma(t) = \frac{\gamma_0}{1 + \gamma_0 \int_0^t \alpha^2(s)ds}, \quad m_t = m_0 + \int_0^t \gamma(s)\alpha(s)dB_s$$

where $\alpha = a_1/b$. Furthermore, $\mathbb{F}^Y = \mathbb{F}^B$.

Filtering versus enlargement: Choosing $f(s) = \frac{\gamma(s)a_1(s)}{b(s)}$ in the example of Section 5.4.4 leads to the same conditional law (with $m_0 = 0$); indeed, it is not difficult to check that this choice of parameter leads to $\int_t^\infty f^2(s)ds = \sigma^2(t) = \gamma(t)$ so that the two variances are equal.

The similarity between filtering and the example of Section 5.4.4 can be also explained as follows. Let us start from the setting of Section 5.4.4 where $X = \int_0^\infty f(s)dB_s$ and introduce $\mathbb{F}^{(X)} = \mathbb{F}^B \vee \sigma(X)$, where B is the given Brownian motion. We have seen that

$$W_t := B_t + \int_0^t \frac{X - m_s}{\sigma^2(s)} f(s)ds$$

is an $\mathbb{F}^{(X)}$-BM, hence is a $\mathbb{G}^W$-BM independent of X. So, the example presented in Section 5.4.4is equivalent to the following filtering problem: the signal X is a Gaussian variable, centered, with variance $\gamma(0) = \int_0^\infty f^2(s)ds$ and the observation

$$dY_t = f(t)Xdt + \left(\int_t^\infty f^2(s)ds \right) dW_t = f(t)Xdt + \sigma^2(t)dW_t \, .$$

6.2.5 Disorder

Classical case: the signal is independent of the driving Brownian motion

Let $W = (W_t)_{t\geq 0}$ be a Brownian motion defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$, and τ be a random time, independent of W and such that $\mathbb{P}(\tau > t) = e^{-\lambda t}$, for all $t \geq 0$ and some $\lambda > 0$ fixed. We define $Y = (Y_t)_{t\geq 0}$ as the solution of the stochastic differential equation

$$dY_t = \big(a + b\,\mathbb{1}_{\{t>\tau\}}\big)\,dt + Y_t\,\sigma\,dW_t\,.$$

Let $\mathbb{F}^Y = (\mathcal{F}^Y_t, t \geq 0)$ be the natural filtration of the process Y (note that $\mathbb{F}^Y$ is smaller than $\mathbb{F}^W \vee \sigma(\tau)$). From $Y_t = x + \int_0^t (a + b\,\mathbb{1}_{\{s>\tau\}})ds + \int_0^t \sigma\,dW_s$, it follows that (from Exercise 1.5.5)

$$dY_t = \,\big(a + b\,(1 - G_t)\big)\,dt + d\text{mart}$$

Here, $G = (G_t)_{t\geq 0}$ is the Azéma supermartingale given by $G_t = \mathbb{P}(\tau > t\,|\,\mathcal{F}_t)$. Identifying the brackets, one has $d\text{mart} = \sigma d\bar{W}_t$ where $\bar{W}$ is a martingale with bracket t, hence is a BM. It follows that the process Y admits the following representation in its own filtration

$$dY_t = \,\big(a + b\,(1 - G_t)\big)\,dt + \,\sigma\,d\bar{W}_t\,.$$

Here $\bar{W} = (\bar{W}_t)_{t\geq 0}$ is the innovation process defined by

$$\bar{W}_t = W_t + \frac{b}{\sigma}\int_0^t \big(\mathbb{1}_{\{s>\tau\}} - (1 - G_s)\big)\,ds = W_t - \frac{b}{\sigma}\int_0^t \big(\mathbb{1}_{\{\tau>s\}} - G_s\big)\,ds$$

and is a standard $\mathbb{F}$-Brownian motion. Using the previous results with $a(t, Y_t, \tau) = a + b\mathbb{1}_{t>\tau}$, one obtains easily

$$\begin{aligned}
\ell_t(u) &= \exp\left(\frac{a}{\sigma^2}Y_t - \frac{1}{2}\frac{a^2}{\sigma^2}t\right) =: Z_t \quad u > t \\
&= \exp\left(\frac{a+b}{\sigma^2}Y_t - \frac{1}{2}\frac{(a+b)^2}{\sigma^2}t - \frac{b}{\sigma^2}Y_u + \frac{1}{2}\Big(\frac{b^2}{\sigma^2} + \frac{2ab}{\sigma^2}\Big)u\right) \\
&= \frac{Z_t}{U_t}U_u \quad u \leq t
\end{aligned}$$

where $U_u = e^{-\frac{b}{\sigma^2}Y_u + \frac{1}{2}(\frac{b^2}{\sigma^2} + \frac{2ab}{\sigma^2})u}$ and $G_t = \frac{1}{m^\ell_t}e^{-\lambda t}Z_t$ where

$$\begin{aligned}
m^\ell_t &= \lambda\,e^{\frac{a+b}{\sigma^2}Y_t - \frac{1}{2}\frac{(a+b)^2}{\sigma^2}t}\int_0^t e^{-\lambda u}e^{-\frac{b}{\sigma^2}Y_u + \frac{1}{2}(\frac{b^2}{\sigma^2} + \frac{2ab}{\sigma^2})u}du + e^{-\lambda t}Z_t \\
&= \lambda\frac{Z_t}{U_t}\int_0^t e^{-\lambda u}U_u du + e^{-\lambda t}Z_t
\end{aligned}$$

Moreover

$$\begin{aligned}
g_t(u) &= \frac{U_t}{e^{-\lambda t}U_t + \lambda\int_0^t e^{-\lambda u}U_u du}\big(\mathbb{1}_{u>t}e^{-\lambda u}U_t + \mathbb{1}_{t>u}U_u\big) \\
G_t(u) &= \frac{Z_t}{m_t}\left(e^{-\lambda t} + \mathbb{1}_{t>u}\frac{1}{U_t}\int_u^t \lambda e^{-\lambda s}U_s ds\right)
\end{aligned}$$

After some computation, we recover that the process G solves the stochastic differential equation

$$dG_t = -\lambda G_t\,dt + \frac{b}{\sigma}\,G_t(1 - G_t)\,d\bar{W}_t\,. \tag{6.2.5}$$

Observe that the process $n = (n_t)_{t\geq 0}$ with $n_t = e^{\lambda t}\, G_t$ admits the representation

$$dn_t = d(e^{\lambda t}\, G_t) = \frac{b}{\sigma}\, e^{\lambda t}\, G_t(1-G_t)\, d\bar{W}_t$$

and thus, n is an $\mathbb{F}$-martingale (to establish the true martingale property, note that the process $(G_t(1-G_t))_{t\geq 0}$ is bounded). The equality (6.2.5) provides the (additive) Doob-Meyer decomposition of the supermartingale G, while $G_t = (G_t\, e^{\lambda t})\, e^{-\lambda t}$ gives its multiplicative decomposition. It follows from these decompositions that the $\mathbb{F}$-intensity rate of τ is λ, so that, the process $M = (M_t)_{t\geq 0}$ with $M_t = \mathbb{1}_{\tau\leq t} - \lambda(t\wedge\tau)$ is a $\mathbb{G}$-martingale.

It follows from the definition of the conditional survival probability process G and the fact that $(G_t\, e^{\lambda t})_{t\geq 0}$ is a martingale that the expression

$$\mathbb{P}(\tau > u \,|\, \mathcal{F}_t) = \mathbb{E}[\mathbb{P}(\tau > u \,|\, \mathcal{F}_u) \,|\, \mathcal{F}_t] = \mathbb{E}[G_u\, e^{\lambda u} \,|\, \mathcal{F}_t]\, e^{-\lambda u} = G_t\, e^{\lambda(t-u)}$$

holds for $0 \leq t < u$. One can easily extend the results to the case

$$dY_t = \big(a(t,Y_t) + b(t,Y_t)\, \mathbb{1}_{t>\tau}\big)\, dt + \sigma(t,Y_t)\, dW_t\,.$$

Using the previous results with $a(t,Y_t,\tau) = a(t,Y_t) + b(t,Y_t)\mathbb{1}_{t>\tau} := a_t + b_t\mathbb{1}_{t>\tau}$, one obtains easily

$$\begin{aligned}
\ell_t(u) &= \exp\left(\int_0^t \frac{a_s}{\sigma_s^2} dY_s - \frac{1}{2}\frac{a_s^2}{\sigma_s^2} ds\right) =: Z_t \quad u > t \\
&= \exp\left(\int_0^u \frac{a_s}{\sigma_s^2} dY_s - \int_0^u \frac{1}{2}\frac{a_s^2}{\sigma_s^2} ds + \int_u^t \frac{a_s+b_s}{\sigma_s^2} dY_s - \int_u^t \frac{1}{2}\frac{(a_s+b_s)^2}{\sigma_s^2} ds\right) \quad u \leq t
\end{aligned}$$

and $G_t = \frac{1}{m_t^\ell} e^{-\lambda t} Z_t$ where

$$\begin{aligned}
m_t^\ell &= \lambda \exp\left(\int_0^t \frac{a_s+b_s}{\sigma_s^2} dY_s - \int_0^t \frac{1}{2}\frac{(a_s+b_s)^2}{\sigma_s^2} ds\right) \int_0^t e^{-\lambda u} U_u du + e^{-\lambda t} Z_t \\
&= \lambda \frac{Z_t}{U_t} \int_0^t e^{-\lambda u} U_u du + e^{-\lambda t} Z_t
\end{aligned}$$

with $U_u = \exp\left(-\int_0^u \frac{b_s}{\sigma_s^2} dY_s + \frac{1}{2}\int_0^u \frac{b_s}{\sigma_s^2} + 2\frac{a_s b_s}{\sigma_s^2} ds\right)$

6.2.6 Detection

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Chapter 7

Progressive Enlargement

In this chapter, we study the case of progressive enlargements of the form $\mathcal{F}_t \vee \sigma(\tau \wedge t)$ for a non negative random variable τ. More precisely, we assume that τ is a finite random time, i.e., a finite non-negative random variable, and we denote by $\mathbb{G}$ the right-continuous filtration

$$\mathcal{G}_t := \cap_{\epsilon>0} \{\mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge (t+\epsilon))\} .$$

We define the right-continuous process H, called the default indicator as

$$H_t = 1\!\!1_{\{\tau \le t\}} .$$

We denote by $\mathbb{H} = (\mathcal{H}_t, t \ge 0)$ its natural filtration (after regularization). With an abuse of notation, we write $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ for the right-continuous progressively enlarged filtration. Note that τ is an $\mathbb{H}$-stopping time, hence a $\mathbb{G}$-stopping time. (In fact, $\mathbb{H}$ is the smallest right-continuous filtration making τ a stopping time, and $\mathbb{G}$ is the smallest right-continuous filtration containing $\mathbb{F}$ and making τ a stopping time).

We recall the result obtained in Subsection 2.2.1: if Y is a $\mathbb{G}$-adapted process, there exists an $\mathbb{F}$-adapted process $Y^{\mathbb{F}}$, called the predefault-value of Y, such that $1\!\!1_{\{t<\tau\}} Y_t = 1\!\!1_{\{t<\tau\}} Y^{\mathbb{F}}_t$.

For a general random time τ, it is not true that $\mathbb{F}$-martingales are $\mathbb{G}$-semi-martingales. Here is an example: due to the separability of the Brownian filtration, there exists a bounded random variable τ such that $\mathcal{F}_\infty = \sigma(\tau)$. Hence, $\mathcal{F}^\tau_{\tau+t} = \mathcal{F}_\infty, \forall t$ so that the $\mathbb{G}$-martingales are constant after τ. Consequently, $\mathbb{F}$-martingales are not $\mathbb{G}$-semi-martingales.

In this chapter, after a presentation of some general results, we pay attention to particular families of pseudo-stopping times and honest times. The study of initial and equivalent times is deferred to the following chapters. The study of the particular and important case of last passage times is presented in Chapter 10.

We recall the two important conditions
(C) All $\mathbb{F}$-martingales are continuous
(A) τ avoids $\mathbb{F}$-stopping times, i.e., $\mathbb{P}(\tau = \vartheta) = 0$ for any $\mathbb{F}$-stopping time ϑ.

We recall our notation

$$\boxed{\mathbb{F} \subset \mathbb{G} = \mathbb{F} \vee \mathbb{H} \subset \mathbb{F}^{(\tau)} = \mathbb{F} \vee \sigma(\tau)}$$

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7.1 An important supermartingale

We introduce the Azéma supermartingale $Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ and call it sometimes the conditional survival process. The process Z is a super-martingale of class (D). Therefore, it admits a Doob-Meyer decomposition.

Lemma 7.1.1 *Let τ be a positive random time and*

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \mu_t - A^p_t$$

the Doob-Meyer decomposition of the super-martingale Z (the process $A^p = A^{p,\mathbb{F}}$ is the $\mathbb{F}$-predictable compensator of H, see Definition 1.4.7). Then, for any $\mathbb{F}$-predictable positive process Y,
(i)

$$\mathbb{E}(Y_\tau) = \mathbb{E}\left(\int_0^\infty Y_u dA^p_u\right).$$

(ii)

$$\mathbb{E}(Y_\tau 1\!\!1_{t<\tau\le T}|\mathcal{F}_t) = \mathbb{E}\left(\int_t^T Y_u dA^p_u|\mathcal{F}_t\right) = -\mathbb{E}\left(\int_t^T Y_u dZ_u|\mathcal{F}_t\right)$$

PROOF: (i) is a consequence of the definition (see Proposition 1.4.8).
For any càglàd process Y of the form $Y_u = y_s 1\!\!1_{]s,t]}(u)$ with $y_s \in b\mathcal{F}_s$, one has

$$\mathbb{E}(Y_\tau) = \mathbb{E}(y_s 1\!\!1_{]s,t]}(\tau)) = \mathbb{E}(y_s(A_t - A_s)).$$

The result follows from MCT. □
The process Z, taken càdlàg is the $\mathbb{F}$-optional projection of $1\!\!1_{\tau>t}$. If $R := \inf\{t : Z_t = 0\}$, then $\mathbb{P}(\tau > R|\mathcal{F}_R)1\!\!1_{R<\infty} = Z_R 1\!\!1_{R<\infty} = 0$. Moreover, $\mathbb{P}(Z_{\tau-} > 0) = 1$.

Anther important $\mathbb{F}$-supermartingale is

$$\widetilde{Z}_t := P\left(\tau \ge t \;\middle|\; \mathcal{F}_t\right). \tag{7.1.1}$$

The supermartingale Z is right-continuous with left limits and coincides with the $\mathbb{F}$-optional projection of $1\!\!1_{]\!]0,\tau[\![}$, while $\widetilde{Z}$ admits right limits and left limits only and is the $\mathbb{F}$-optional projection of $1\!\!1_{]\!]0,\tau]\!]}$. An optional decomposition of Z leads to an important $\mathbb{F}$-martingale m, given by

$$m := Z + A^{o,\mathbb{F}}, \tag{7.1.2}$$

where $A^{o,\mathbb{F}}$ is the $\mathbb{F}$-dual optional projection of H. The supermartingales Z and e Z are related through $\widetilde{Z} = Z + \Delta A^{o,\mathbb{F}}$. If assumption (C) or (A) is satisfies, then $Z = \widetilde{Z}$. Under assumptions (C) and (A), the supermartingale $Z = \widetilde{Z}$ is a continuous process.

Proposition 7.1.2 *The process μ is a square integrable martingale*

PROOF: From Doob-Meyer decomposition, since Z is bounded, μ is a square integrable martingale.□

7.2 General Facts

For what concerns the progressive enlargement setting, the following result is analogous to Proposition 5.1.1. This results can be found in Jeulin [73, Lemma 4.4].

Proposition 7.2.1 *One has*

(i) A random variable Y_t is $\mathcal{G}_t$-measurable if and only if it is of the form

$$Y_t(\omega) = \tilde{y}_t(\omega)\mathbb{1}_{t<\tau(\omega)} + \hat{y}_t(\omega, \tau(\omega))\mathbb{1}_{\tau(\omega)\le t}$$

for some $\mathcal{F}_t$-measurable random variable $\tilde{y}_t$ and some family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variables $\hat{y}_t(\cdot, u), t \ge u$.

(ii) A process Y is $\mathbb{G}$-predictable if and only if it is of the form

$$Y_t(\omega) = \tilde{y}_t(\omega)\mathbb{1}_{t\le\tau(\omega)} + \hat{y}_t(\omega, \tau(\omega))\mathbb{1}_{\tau(\omega)<t}, t \ge 0,$$

where $\tilde{y}$ is $\mathbb{F}$-predictable and $(t, \omega, u) \mapsto \hat{y}_t(\omega, u)$ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function.

PROOF: For part (i), it suffices to recall that $\mathcal{G}_t$-measurable random variables are generated by random variables of the form $X_t(\omega) = x_t(\omega)f\big(t \wedge \tau(\omega)\big)$, with $x_t \in \mathcal{F}_t$ and f a bounded Borel function on $\mathbb{R}^+$.

(ii) It suffices to notice that $\mathbb{G}$-predictable processes are generated by processes of the form $X_t = x_t\mathbb{1}_{t\le\tau} + \hat{x}_t f(\tau)\mathbb{1}_{\tau<t}$, $t \ge 0$, where $x, \hat{x}$ are $\mathbb{F}$-predictable and f is a bounded Borel function, defined on $\mathbb{R}^+$. □

Such a characterization result does *not* hold for optional processes, in general. We refer to Barlow [15, Remark on pages 318 and 319], for a counterexample. See Song [106] for a general study.

7.2.1 A Larger Filtration

We also introduce the filtration $\widetilde{\mathbb{F}}^\tau$ defined as

$$\widetilde{\mathcal{F}}_t^\tau = \{A \in \mathcal{F}_\infty \vee \sigma(\tau) \,:\, \exists \widetilde{A}_t \in \mathcal{F}_t \,:\, A \cap \{\tau > t\} = \widetilde{A}_t \cap \{\tau > t\}\,\}$$

Lemma 7.2.2 *Let ϑ be a $\widetilde{\mathbb{F}}^\tau$-stopping time, then, there exists an $\mathbb{F}$-stopping time $\widetilde{\vartheta}$ such that $\vartheta \wedge \tau = \widetilde{\vartheta} \wedge \tau$*
Let $X \in \mathcal{F}_\infty$. A càdlàg version of $M_t^X := \mathbb{E}(X|\widetilde{\mathcal{F}}_t^\tau)$ is

$$\mathbb{1}_{t<\tau}\frac{1}{\mathbb{P}(\tau > t|\mathcal{F}_t)}\mathbb{E}(X\mathbb{1}_{t<\tau}|\mathcal{F}_t) + X\mathbb{1}_{t\ge\tau}$$

and

$$M_{t-}^X = \frac{1}{Z_{t-}}\,{}^p(X\mathbb{1}_{]\!]0,\tau]\!]}) + X\mathbb{1}_{t>\tau}$$

7.2.2 Conditional Expectations

Proposition 7.2.3 *For any $\mathbb{G}$-predictable process Y, there exists an $\mathbb{F}$-predictable process y such that $Y_t\mathbb{1}_{\{t\le\tau\}} = y_t\mathbb{1}_{\{t\le\tau\}}$. Under the condition $\forall t, \mathbb{P}(\tau \le t|\mathcal{F}_t) < 1$, the process $(y_t, t \ge 0)$ is unique.*

PROOF: We refer to Dellacherie [39] and Dellacherie et al. [35, p.186]. The process y may be recovered as the ratio of the $\mathbb{F}$-predictable projections of $Y_t\mathbb{1}_{\{t\le\tau\}}$ and $\mathbb{1}_{\{t\le\tau\}}$. □
✓PRECISER

Lemma 7.2.4 ***Key Lemma:*** *Let $X \in \mathcal{F}_T$ be an integrable r.v. Then, for any $t \le T$,*

$$\mathbb{E}(X\mathbb{1}_{\{\tau<T\}}|\mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}}\frac{\mathbb{E}(XZ_T|\mathcal{F}_t)}{Z_t}$$

PROOF: On the set $\{t < \tau\}$, any $\mathcal{G}_t$ measurable random variable is equal to an $\mathcal{F}_t$-measurable random variable, therefore

$$\mathbb{E}(X 1\!\!1_{\{\tau<T\}}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}} y_t$$

where y_t is $\mathcal{F}_t$-measurable. Taking conditional expectation w.r.t. $\mathcal{F}_t$, we get $y_t = \dfrac{\mathbb{E}(Y_t 1\!\!1_{\{t<\tau\}}|\mathcal{F}_t)}{\mathbb{P}(t<\tau|\mathcal{F}_t)}$. (it can be proved that $\mathbb{P}(t < \tau|\mathcal{F}_t)$ does not vanish on the set $\{t < \tau\}$, see the following Exercise 7.2.6.) □

Exercise 7.2.5 Prove that, if τ is an $\mathbb{F}$ stopping time, $\mathbb{G} = \mathbb{F}$. ◁

Exercise 7.2.6 Prove that

$$\{\tau > t\} \subset \{Z_t > 0\} \tag{7.2.1}$$

(where the inclusion is up to a negligible set). ◁

7.3 Before τ

It is proved in Yor [113] that, if X is an $\mathbb{F}$-martingale, then the processes $X_{t\wedge\tau}$ and $X_t(1 - H_t)$ are $\mathbb{G}$ semi-martingales. Furthermore, the decompositions of the $\mathbb{F}$-martingales in the filtration $\mathbb{G}$ are known up to time τ (Jeulin and Yor [74]).

Proposition 7.3.1 *Under (CA), every $\mathbb{F}$-martingale X stopped at time τ is a $\mathbb{G}$-semi-martingale with canonical decomposition*

$$X_t^\tau = X_t^{\mathbb{G}} + \int_0^{t\wedge\tau} \frac{d\langle X,\mu\rangle_s}{Z_s}\,,$$

where $X^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale.

PROOF: Let Y_s be an $\mathcal{G}_s$-measurable random variable. There exists an $\mathcal{F}_s$-measurable random variable y_s such that $Y_s 1\!\!1_{\{s<\tau\}} = y_s 1\!\!1_{\{s<\tau\}}$, hence, if X is an $\mathbb{F}$-martingale, for $s < t$,

$$\begin{aligned}
&\mathbb{E}(Y_s(X_{t\wedge\tau} - X_{s\wedge\tau})) = \mathbb{E}(Y_s 1\!\!1_{\{s<\tau\}}(X_{t\wedge\tau} - X_{s\wedge\tau}))\\
&= \mathbb{E}(y_s 1\!\!1_{\{s<\tau\}}(X_{t\wedge\tau} - X_{s\wedge\tau}))\\
&= \mathbb{E}\left(y_s(1\!\!1_{\{s<\tau\le t\}}(X_\tau - X_s) + 1\!\!1_{\{t<\tau\}}(X_t - X_s))\right)
\end{aligned}$$

From the definition of Z (see also Definition 1.4.7 and Lemma 7.1.1),

$$\mathbb{E}\left(y_s 1\!\!1_{\{s<\tau\le t\}} X_\tau\right) = -\mathbb{E}\left(y_s \int_s^t X_u dZ_u\right).$$

From integration by parts formula

$$\int_s^t X_u dZ_u = -X_s Z_s + Z_t X_t - \int_s^t Z_u dX_u - \langle X,Z\rangle_t + \langle X,Z\rangle_s$$

We have also

$$\begin{aligned}
\mathbb{E}\left(y_s 1\!\!1_{\{s<\tau\le t\}} X_s\right) &= \mathbb{E}\left(y_s X_s (Z_s - Z_t)\right)\\
\mathbb{E}\left(y_s 1\!\!1_{\{t<\tau\}}(X_t - X_s)\right) &= \mathbb{E}\left(y_s Z_t (X_t - X_s)\right)
\end{aligned}$$

hence, from the martingale property of X

$$\begin{aligned}
&\mathbb{E}(Y_s(X_{t\wedge\tau}-X_{s\wedge\tau}))=\mathbb{E}(y_s(\langle X,\mu\rangle_t-\langle X,\mu\rangle_s))\\
&= \mathbb{E}\left(y_s\int_s^t\frac{d\langle X,\mu\rangle_u}{Z_u}Z_u\right)=\mathbb{E}\left(y_s\int_s^t\frac{d\langle X,\mu\rangle_u}{Z_u}\mathbb{E}(\mathbb{1}_{\{u<\tau\}}|\mathcal{F}_u)\right)\\
&= \mathbb{E}\left(y_s\int_s^t\frac{d\langle X,\mu\rangle_u}{Z_u^\tau}\mathbb{1}_{\{u<\tau\}}\right)=\mathbb{E}\left(y_s\int_{s\wedge\tau}^{t\wedge\tau}\frac{d\langle X,\mu\rangle_u}{Z_u}\right).
\end{aligned}$$

The result follows. □

In the general case, denoting by $A^{(o,\mathbb{F})}$ the dual optional projection of $\mathbb{1}_{t\geq\tau}$, and m the martingale $m:=Z+A^{(o,\mathbb{F})}$, every $\mathbb{F}$-martingale X stopped at time τ is a $\mathbb{G}$-semi-martingale with canonical decomposition

$$X_t^\tau=X_t^{\mathbb{G}}+\int_0^{t\wedge\tau}\frac{d\langle X,m\rangle_s}{Z_{s-}},$$

where $\widetilde{X}$ is a $\mathbb{G}$-local martingale. Note that $m_t=\mathbb{E}(A^o_\infty|\mathcal{F}_t)$ and, for any $\mathbb{F}$ uniformly integrable martingale n, $E(n_\tau)=\mathbb{E}(n_\infty m_\infty)$. In other terms,

$$X_{t\wedge\tau}=X_t^{\mathbb{G}}+\int_0^{t\wedge\tau}\frac{d\langle X,\mu\rangle_s+dJ_s}{Z_{s-}},$$

where J is the dual predictable projection of $\Delta X_\tau\mathbb{1}_{[\![\tau,\infty[\![}$.

Another interesting decomposition is (see Aksamit [1]) Let us introduce the $\mathbb{F}$-stopping time R $R:=\inf\{t: Z_t=0\}$ and $\widetilde{R}=R_{\{\widetilde{Z}_R=0<Z_{R-}\}}$ Then, if X is an $\mathbb{F}$-local martingale, the process

$$X_t^\tau-\int_0^{t\wedge\tau}\frac{1}{\widetilde{Z}_s}d[m,X]_s+(\Delta X_{\widetilde{R}}\,\mathbb{1}_{[\![\widetilde{R},\infty[\![})^p_{t\wedge\tau}$$

is a $\mathbb{G}$ martingale.

This result remains valid for any filtration $\mathbb{G}$ that coincide with $\mathbb{F}$ before τ.

7.4 Basic Results

We recall the results obtained in Proposition 2.2.8:

Proposition 7.4.1 *The process*

$$M_t=H_t-\int_0^{t\wedge\tau}\frac{dA_u^\tau}{Z_{u-}}$$

is a $\mathbb{G}$-martingale.

For any bounded $\mathbb{G}$-predictable process Y, the process

$$Y_\tau\mathbb{1}_{\tau\leq t}-\int_0^{t\wedge\tau}\frac{Y_s}{Z_{s-}^\tau}dA_s^\tau$$

is a $\mathbb{G}$-martingale.
The process $L_t:=(1-H_t)/G_t$ is a $\mathbb{G}$ -martingale.

Definition 7.4.2 *In the case where the process A is absolutely continuous w.r.t. Lebesgue's measure, i.e., $dA_u^\tau=a_u du$, the process $\lambda_t=\frac{a_t}{Z_{t-}}$ is called the $\mathbb{F}$-intensity of τ, the process $\lambda_t^{\mathbb{G}}=\mathbb{1}_{t<\tau}\lambda_t$ is the $\mathbb{G}$-intensity, and the process*

$$H_t-\int_0^{t\wedge\tau}\lambda_s ds=H_t-\int_0^t(1-H_s)\lambda_s ds=H_t-\int_0^t\lambda_s^{\mathbb{G}}ds$$

is a $\mathbb{G}$-martingale.

The intensity process is the basic tool to model default risk.

We also recall

Lemma 7.4.3 *The process λ satisfies*

$$\lambda_t = \lim_{h\to 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t+h|\mathcal{F}_t)}{\mathbb{P}(t < \tau|\mathcal{F}_t)} .$$

The converse is known as Aven's lemma [10].

Lemma 7.4.4 *Let $(\Omega, \mathbb{G}, \mathbb{P})$ be a filtered probability space and N be a counting process. Assume that $E(N_t) < \infty$ for any t. Let $(h_n, n \geq 1)$ be a sequence of real numbers converging to 0, and*

$$Y_t^{(n)} = \frac{1}{h_n} E(N_{t+h_n} - N_t|\mathcal{G}_t)$$

Assume that there exists λ and y non-negative $\mathbb{G}$-adapted processes such that

(i) For any t, $\lim Y_t^{(n)} = \lambda_t$

(ii) For any t, there exists for almost all ω an $n_0 = n_0(t,\omega)$ such that

$$|Y_s^{(n)}(\omega) - \lambda_s(\omega)| \leq y_s(\omega)\, ,\; s \leq t,\, n \geq n_0(t,\omega)$$

(iii) $\int_0^t y_s ds < \infty, \forall t, a.s.$

Then, $N_t - \int_0^t \lambda_s ds$ is a $\mathbb{G}$-martingale.

Exercise 7.4.5 Prove that if X is a (square-integrable) $\mathbb{F}$-martingale, XL is a $\mathbb{G}$ -martingale, where L is defined in Proposition 7.4.1. ◁

Exercise 7.4.6 We consider, as in the paper of Biagini et al. [19] a mortality bond, a financial instrument with payoff $Y = \int_0^{\tau\wedge T} Z_s ds$, where $Z_s = \mathbb{P}(\tau > s|\mathcal{F}_s)$ where $\mathbb{F}$ is a continuous filtration. We assume that Z is continuous, admits a Doob-Meyer decomposition as $Z = \mu - A$ and does not vanish.

1. Compute, in the case $r = 0$, the price Y_t of the mortality bond. It will be convenient to introduce $N_t = \mathbb{E}(\int_0^T Z_s^2 ds|\mathcal{F}_t)$. Is the process N a $(\mathbb{P},\mathbb{F})$ martingale? a $(\mathbb{P},\mathbb{G})$-martingale?
2. Determine the processes α, β and γ so that
$$dY_t = \alpha_t dM_t + \beta_t (dN_t - \frac{1}{Z_t} d\langle N, Z\rangle_t) + \gamma_t (dZ_t - \frac{1}{Z_t} d\langle Z\rangle_t)$$
3. Determine the price $D(t,T)$ of a defaultable zero-coupon bond with maturity T, i.e., a financial asset with terminal payoff $1\!\!1_{T<\tau}$. Give the dynamics of this price.
4. We now assume that $\mathbb{F}$ is a Brownian filtration, and that a risky asset with dynamics
$$dS_t = S_t(bdt + \sigma dW_t)$$
is traded. Explain how one can hedge the mortality bond.

◁

7.4.1 Restricting the information

Suppose from now on that a second filtration $\widetilde{\mathbb{F}}$ is given, with $\widetilde{\mathcal{F}}_t \subset \mathcal{F}_t$ and define the associated σ-algebra $\widetilde{\mathcal{G}}_t = \widetilde{\mathcal{F}}_t \vee \mathcal{H}_t$ and the Azéma super-martingale

$$\widetilde{Z}_t = \mathbb{P}(t < \tau | \widetilde{\mathcal{F}}_t) = \mathbb{E}(Z_t | \widetilde{\mathcal{F}}_t) .$$

Let $Z_t = \mu_t - A^p_t$ be the $\mathbb{F}$-Doob-Meyer decomposition of the $\mathbb{F}$-supermartingale Z and assume that A^p is absolutely continuous with respect to Lebesgue's measure: $A^p_t = \int_0^t a_s ds$. The process $\widetilde{A}_t = \mathbb{E}(A_t | \widetilde{\mathcal{F}}_t)$ is an $\widetilde{\mathbb{F}}$-submartingale and its $\widetilde{\mathbb{F}}$-Doob-Meyer decomposition is denoted

$$\widetilde{A}_t = \widetilde{n}_t + \widetilde{\alpha}_t .$$

where $\widetilde{n}$ is the martingale part. Hence, setting $\widetilde{\mu}_t = \mathbb{E}(\mu_t | \widetilde{\mathcal{F}}_t)$, the super-martingale $\widetilde{Z}$ admits a $\widetilde{\mathbb{F}}$-Doob-Meyer decomposition as

$$\widetilde{Z}_t = \widetilde{\mu}_t - \widetilde{n}_t - \widetilde{\alpha}_t$$

where $\widetilde{\mu}_t - \widetilde{n}_t$ is the martingale part. From Exercise 1.5.5, $\widetilde{\alpha}_t = \int_0^t \mathbb{E}(a_s | \widetilde{\mathcal{F}}_s) ds$. It follows that

$$H_t - \int_0^{t \wedge \tau} \frac{d\widetilde{\alpha}_s}{\widetilde{Z}_s} ds = H_t - \int_0^{t \wedge \tau} \frac{\mathbb{E}(a_s | \widetilde{\mathcal{F}}_s)}{\widetilde{Z}_s} ds$$

is a $\widetilde{\mathbb{G}}$-martingale and that the $\widetilde{\mathbb{F}}$-intensity of τ is equal to $\mathbb{E}(a_s | \widetilde{\mathcal{F}}_s)/\widetilde{Z}_s$, and not "as one could think" to $\mathbb{E}(a_s/Z_s | \widetilde{\mathcal{F}}_s)$.

This result can be directly proved thanks to Brémaud's following result (a consequence of Exercise 1.5.5): if $H_t - \int_0^t \widetilde{\lambda}_s ds$ is a $\mathbb{G}$-martingale, then $H_t - \int_0^t \mathbb{E}(\widetilde{\lambda}_s | \widetilde{\mathcal{G}}_s) ds$ is a $\widetilde{\mathbb{G}}$-martingale. Since

$$\begin{aligned} \mathbb{E}(\widetilde{X}_s | \mathcal{F}_s) = \mathbb{E}(\mathbb{1}_{\{s \le \tau\}} \lambda_s | \widetilde{\mathcal{G}}_s) &= \frac{\mathbb{1}_{\{s \le \tau\}}}{\widetilde{Z}_s} \mathbb{E}(\mathbb{1}_{\{s \le \tau\}} \lambda_s | \widetilde{\mathcal{F}}_s) \\ &= \frac{\mathbb{1}_{\{s \le \tau\}}}{\widetilde{Z}_s} \mathbb{E}(Z_s \lambda_s | \widetilde{\mathcal{F}}_s) = \frac{\mathbb{1}_{\{s \le \tau\}}}{\widetilde{Z}_s} \mathbb{E}(a_s | \widetilde{\mathcal{F}}_s) \end{aligned}$$

it follows that $H_t - \int_0^{t \wedge \tau} \mathbb{E}(a_s | \widetilde{\mathcal{F}}_s)/\widetilde{Z}_s ds$ is a $\widetilde{\mathbb{G}}$-martingale, and we are done.

7.4.2 Multiplicative Decomposition of the Survival Process

✓Voir survey Ashkan, faire cas general (sans continuité

Lemma 7.4.7 *Assume that the super-martingale G does not vanish. Then, G admits a multiplicative decomposition as $Z_t = N_t D_t$ where D is a decreasing $\mathbb{F}$-predictable process and N a local $\mathbb{F}$-martingale.*

Proof: In the proof, we assume that Z is continuous. Assuming that the supermartingale $Z = \mu - A$ admits a multiplicative decomposition $Z_t = D_t N_t$ where N is a local-martingale, assumed to be continuous and D a decreasing process, we obtain that $dZ_t = N_t dD_t + D_t dN_t$ is a Doob-Meyer decomposition of Z. Hence,

$$dN_t = \frac{1}{D_t} d\mu_t, \; dD_t = -D_t \frac{1}{Z_t} dA_t \, ,$$

therefore,

$$D_t = \exp - \int_0^t \frac{1}{Z_s} dA_s$$

Setting $d\Lambda_t = \frac{dA_t}{Z_t}, \Lambda_0)0$, we obtain $D_t = e^{-\Lambda_t}$, and $H_t - \Lambda_{t \wedge \tau}$ is a $\mathbb{G}$ martingale.
Conversely, if Z admits the Doob-Meyer decomposition $\mu - A$, then $Z_t = N_t D_t$ with $D_t = e^{-\Lambda_t}$, where Λ is the intensity process $\Lambda_t = \int_0^t \frac{1}{Z_s} dA_s$. □

7.5 Pseudo-stopping Times

✓TO BE COMPLETED

As we have mentioned, if $\mathbb{F}$ is immersed in $\mathbb{G}$, the process $(Z_t, t \geq 0)$ is a decreasing process. The converse is not true. The decreasing property of Z is closely related with the definition of pseudo-stopping times, a notion developed from D. Williams example (see Example 7.5.3 below).

Definition 7.5.1 *A random time τ is a pseudo-stopping time if, for any bounded $\mathbb{F}$-martingale M, $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$.*

Proposition 7.5.2 *The random time τ is a pseudo-stopping time if and only if one of the following equivalent properties holds:*
(i) For any local $\mathbb{F}$-martingale M, the process $(M_{t\wedge\tau}, t \geq 0)$ is a local $\mathbb{G}$-martingale
(ii)$A^p_\infty = 1$,
(iii) $\mu_t = 1$, $\forall t \geq 0$,
(iv) The process Z is a decreasing $\mathbb{F}$-predictable process.

Proof: The implication $(iv) \Rightarrow (i)$ is a consequence of Jeulin result established in Theorem 7.3.1. The implication $(i) \Rightarrow (ii)$ follows from the properties of the compensator A^τ: indeed

$$\mathbb{E}(M_\tau) = \mathbb{E}(\int_0^\infty M_u dA^p_u) = \mathbb{E}(M_\infty A^p_\infty) = m_0$$

implies that $A^p_\infty = 1$. We refer to Nikeghbali and Yor [98]. □

Example 7.5.3 The first example of a pseudo-stopping time was given by Williams [110]. Let B be a Brownian motion and define the stopping time $T_1 = \inf\{t : B_t = 1\}$ and the random time $\vartheta = \sup\{t < T_1 : B_t = 0\}$. Set

$$\tau = \sup\{s < \theta : B_s = B^*_s\}$$

where B^* is the running maximum of the Brownian motion. Then, τ is a pseudo-stopping time. Note that $\mathbb{E}(B_\tau)$ is not equal to 0; this illustrates the fact we cannot take any martingale in Definition 7.5.1. The martingale $(B_{t\wedge T_1}, t \geq 0)$ is neither bounded, nor uniformly integrable. In fact, since the maximum B^*_θ $(=B_\tau)$ is uniformly distributed on $[0, 1]$, one has $\mathbb{E}(B_\tau) = 1/2$.

✓Study the stability of pseudo-s times under a change of probability

7.6 Honest Times

There exists an interesting class of random times τ such that $\mathbb{F}$-martingales are $\mathbb{G}$-semi-martingales.

7.6.1 Definition

Definition 7.6.1 *A random time τ is honest if for $s \leq t$*

$$\{\tau \leq s\} = F_{s,t} \cap \{\tau \leq t\}, \textit{ for some } F_{st} \in \mathcal{F}_t$$

or equivalently, if τ is equal to an $\mathcal{F}_t$-measurable random variable on $\tau < t$.

Example 7.6.2 (i) Let t fixed and $g_t = \sup\{s < t : B_s = 0\}$ and set $\tau = g_1$. Then, $g_1 = g_t$ on $\{g_1 < t\}$, and g_t is $\mathcal{F}_t$-measurable.
(ii) Let X be an adapted continuous process and $X^* = \sup X_s, X^*_t = \sup_{s\leq t} X_s$. The random time

$$\tau = \inf\{s : X_s = X^*\}$$

is honest. Indeed, on the set $\{\tau < t\}$, one has $\tau = \inf\{s : X_s = X_s^*\}$.
(iii) An $\mathbb{F}$-stopping time is honest: indeed $\tau = \tau \wedge t$ on $\tau < t$.

If τ is honest,

$$\mathcal{G}_t = \{A \in \mathcal{F}_\infty, : A = (\widetilde{A}_t \cap \{\tau \le t\}) \cup (\widehat{A}_t \cap \{\tau > t\}) \text{ for some } \widehat{A}_t, \widetilde{A}_t \in \mathcal{F}_t\}$$

This filtration is continuous on right.

✓COMPLETER CI DESSOUS

Proposition 7.6.3 *An honest time satisfies* $(\mathcal{H}')$ *hypothesis.*

Proposition 7.6.4 *An honest time does not admit density*

PROOF: For an honest time, $Z_t = \frac{N_t}{N_t^*}$, and N^* is not absolutely continuous. □
See also carthage

Exercise 7.6.5 Let τ be an honest time. Prove that

$$\mathbb{E}(f(\tau)|\mathcal{F}_t) = f(\tau)(1 - Z_t) + \mathbb{E}(\int_t^\infty f(s) dA_s^\tau | \mathcal{F}_t)$$

◁

Exercise 7.6.6 Prove that $\mathcal{G}_t^* := \{A \in \mathcal{F}_\infty : A = (\widetilde{A}_t \cap \{\tau \le t\}) \cup (\widehat{A}_t \cap \{\tau > t\}) \text{ for some } \widehat{A}_t, \widetilde{A}_t \in \mathcal{F}_t\}$ defines indeed a filtration (i.e., the increasing property holds). ◁

7.6.2 Projections of H

✓To be done

7.6.3 Martingales

Proposition 7.6.7 *Let X be a càdlàg $\mathbb{G}$-adapted integrable process. Then X is a $\mathbb{G}$ martingale if and only if*
(i) $\mathbb{E}(X_t|\mathcal{F}_t)$ is an $\mathbb{F}$-martingale
(ii) $\mathbb{E}(\mathbb{1}_{\tau \le s} X_t | \mathcal{F}_s) = \mathbb{E}(\mathbb{1}_{\tau \le s} X_s | \mathcal{F}_s)$.

PROOF: ✓TO BE DONE □

7.6.4 Properties

Proposition 7.6.8 *(Jeulin [73]) A random time τ is honest if it is the end of a predictable set, i.e., $\tau(\omega) = \sup\{t : (t, \omega) \in \Gamma\}$, where Γ is an $\mathbb{F}$-predictable set.*

In particular, an honest time is $\mathcal{F}_\infty$-measurable. If X is a transient diffusion, the last passage time Λ_a (see Proposition 10.1.1) is honest.

✓STABILITY BY SUP Si τ est un temps honnête, les $\mathbb{F}$ martingales restent des martingales après τ. Réciproque? Donner un exemple.

Lemma 7.6.9 *The process Y is $\mathbb{G}$-predictable if and only if there exist two $\mathbb{F}$ predictable processes y and $\widetilde{y}$ such that*

$$Y_t = y_t 1\!\!1_{t\leq\tau} + \widetilde{y}_t 1\!\!1_{t>\tau}.$$

Let $X \in L^1$. Then a càdlàg version of the martingale $X_t = \mathbb{E}\left[X|\mathcal{G}_t\right]$ is given by:

$$X_t = \frac{1}{Z_t^\tau}\mathbb{E}\left[\xi \mathbf{1}_{t<\tau}|\mathcal{F}_t\right]\mathbf{1}_{t<\tau} + \frac{1}{1-Z_t^\tau}\mathbb{E}\left[\xi\mathbf{1}_{t\geq\tau}|\mathcal{F}_t\right]\mathbf{1}_{t\geq\tau}.$$

Every $\mathbb{G}$ optional process decomposes as

$$L1\!\!1_{[0,\tau[} + J1\!\!1_{[\tau]} + K1\!\!1_{]\tau,\infty[},$$

where L and K are $\mathbb{F}$-optional processes and where J is a $\mathbb{F}$ progressively measurable process.

See Jeulin [73] for a proof.

✓Give Barlow's counter example

✓decomposition de Barlow

Lemma 7.6.10 *(Azéma) Let τ be an honest time which avoids $\mathbb{F}$-stopping times. Then:*
(i) A^p_∞ has an exponential law with parameter 1.
(ii) The measure dA^p_t is carried by $\{t\,:\,Z_t = 1\}$
(iii) $\tau = sup\{t\,:\,1-Z_t = 1\}$
(iv)$A^p_\infty = A^p_\tau$

In particular, $\Lambda_t = \int \frac{dA^p}{Z} = A^p_t$ and Λ_τ has an exponential law

7.6.5 Progressive versus initial enlargement

Proposition 7.6.11 *If τ is honest, any $\mathbb{F}$ martingale is a $\mathbb{F}^{(\tau)}$-semi-martingale*

PROOF: ✓SEE JEULIN. TO BE DONE □

✓Question: study the case where (A) is not satisfied

7.6.6 Decomposition

Proposition 7.6.12 *Let τ be honest. We assume that τ avoids stopping times. Then, any $\mathbb{F}$-local martingale M is a $\mathbb{G}$ semi-martingale with decomposition*

$$M_t = \widetilde{M}_t + \int_0^{t\wedge\tau}\frac{d\langle M,\mu\rangle_s}{Z_{s-}} - \int_\tau^{\tau\vee t}\frac{d\langle M,\mu\rangle_s}{1-Z_{s-}},$$

where $\widetilde{M}$ such that is a $\mathbb{G}$-local martingale

PROOF: Let M be an $\mathbb{F}$-martingale which belongs to $\mathbb{H}^1$ and $G_s \in \mathcal{G}_s$. We define a $\mathbb{G}$-predictable process Y as $Y_u = 1\!\!1_{G_s}1\!\!1_{]s,t]}(u)$. For $s < t$, one has, using the decomposition of $\mathbb{G}$-predictable processes:

$$\begin{aligned}\mathbb{E}(1\!\!1_{G_s}(M_t - M_s)) &= \mathbb{E}\left(\int_0^\infty Y_u dM_u\right)\\ &= \mathbb{E}\left(\int_0^\tau y_u dM_u\right) + \mathbb{E}\left(\int_\tau^\infty \widetilde{y}_u dM_u\right).\end{aligned}$$

Noting that $\int_0^t \widetilde{y}_u dM_u$ is a martingale yields $\mathbb{E}\left(\int_0^\infty \widetilde{y}_u dM_u\right) = 0$,

$$\begin{aligned}\mathbb{E}(\mathbb{1}_{G_s}(M_t - M_s)) &= \mathbb{E}\left(\int_0^\tau (y_u - \widetilde{y}_u) dM_u\right) \\ &= \mathbb{E}\left(\int_0^\infty dA_v^p \int_0^v (y_u - \widetilde{y}_u) dM_u\right).\end{aligned}$$

By integration by parts, setting $N_t = \int_0^t (y_u - \widetilde{y}_u) dM_u$, we get

$$\mathbb{E}(\mathbb{1}_{G_s}(M_t - M_s)) = \mathbb{E}(N_\infty A_\infty^p) = \mathbb{E}(N_\infty \mu_\infty) = \mathbb{E}\left(\int_0^\infty (y_u - \widetilde{y}_u) d\langle M, \mu\rangle_u\right).$$

Now, it remains to note that

$$\begin{aligned}&\mathbb{E}\left(\int_0^\infty Y_u \left(\frac{d\langle M,\mu\rangle_u}{Z_{u-}} \mathbb{1}_{\{u\le\tau\}} - \frac{d\langle M,\mu\rangle_u}{1-Z_{u-}} \mathbb{1}_{\{u>\tau\}}\right)\right) \\ &= \mathbb{E}\left(\int_0^\infty \left(y_u \frac{d\langle M,\mu\rangle_u}{Z_{u-}} \mathbb{1}_{\{u\le\tau\}} - \widetilde{y}_u \frac{d\langle M,\mu\rangle_u}{1-Z_{u-}} \mathbb{1}_{\{u>\tau\}}\right)\right) \\ &= \mathbb{E}\left(\int_0^\infty (y_u d\langle M,\mu\rangle_u - \widetilde{y}_u d\langle M,\mu\rangle_u)\right) \\ &= \mathbb{E}\left(\int_0^\infty (y_u - \widetilde{y}_u)\, d\langle M,\mu\rangle_u\right)\end{aligned}$$

to conclude the result in the case $M \in \mathbb{H}^1$. The general result follows by localization. □

Example 7.6.13 Let W be a Brownian motion, and $\tau = g_1$, the last time when the BM reaches 0 before time 1, i.e., $\tau = \sup\{t \le 1 : W_t = 0\}$. Using the computation of $Z_t^{g_1} = \mathbb{P}(g_1 > t|\mathcal{F}_t)$ (see the following Subsection 10.3) and applying Proposition 7.6.12, we obtain the decomposition of the Brownian motion in the enlarged filtration

$$\begin{aligned}W_t &= \widetilde{W}_t - \int_0^t \mathbb{1}_{[0,\tau]}(s) \frac{\Phi'}{1-\Phi}\left(\frac{|W_s|}{\sqrt{1-s}}\right) \frac{\text{sgn}(W_s)}{\sqrt{1-s}}\, ds \\ &\quad + \mathbb{1}_{\{\tau \le t\}} \text{sgn}(W_1) \int_\tau^t \frac{\Phi'}{\Phi}\left(\frac{|W_s|}{\sqrt{1-s}}\right) ds\end{aligned}$$

where $\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x \exp(-u^2/2) du$.

Exercise 7.6.14 Prove that any $\mathbb{F}$-stopping time is honest ◁

Exercise 7.6.15 Prove that

$$\mathbb{E}(\int_0^{t\wedge\tau} \frac{d\langle M,\mu\rangle_s}{Z_{s-}} - \int_\tau^{\tau\vee t} \frac{d\langle M,\mu\rangle_s}{1-Z_{s-}} | \mathcal{F}_t)$$

is an $\mathbb{F}$-local martingale. ◁

7.6.7 Predictable Representation Theorem

Theorem 7.6.16 *If there exists a family of continuous $\mathbb{F}$ martingales M^i which enjoys the PRT in $\mathbb{F}$, then any continuous $\mathbb{G}$-martingale is a sum of stochastic integrals w.r.t. $\widetilde{M^i}$.*

✓ See Barlow

7.6.8 Arbitrages

✓ There are arbitrages "after" τ (Zwieb, Imkeller, Fontana) and before (Aksamit-J)

Tous les spéculateurs cherchent à connaitre L sans jamais y parvenir, d'où son nom de v.a. honnête

(L is here the time where the process reaches it maximum over a time interval)

7.6.9 Stability

Let τ and τ^* be two honest times. We show in the following lemma that $\tau \vee \tau^*$ and $\tau \wedge \tau^*$ are again honest times.

Lemma 7.6.17 *Let τ and τ^* be two honest times, then $\tau \vee \tau^*$ and $\tau \wedge \tau^*$ are also honest times.*

Proof: The random time τ and τ^* are honest, this implies that for every $t \geq 0$ there exist $\mathcal{F}_t$ measurable random variables τ_t and τ_t^* such that

$$\tau \mathbb{1}_{\tau<t} = \tau_t \mathbb{1}_{\tau<t} \quad \text{and} \quad \tau^* \mathbb{1}_{\tau^*<t} = \tau_t^* \mathbb{1}_{\tau^*<t}$$

holds. Let us first consider the random time $\tau \vee \tau^*$.

$$\tau \vee \tau^* \mathbb{1}_{\tau\vee\tau^*<t} = \tau \vee \tau^* \mathbb{1}_{\tau<t,\tau^*<t} = \tau_t \vee \tau_t^* \mathbb{1}_{\tau<t,\tau^*<t} = \tau_t \vee \tau_t^* \mathbb{1}_{\tau\vee\tau^*<t},$$

which proves that it is in fact honest time. On the other hand, for the random time $\tau \wedge \tau^*$,

$$\begin{aligned} \tau \wedge \tau^* \mathbb{1}_{\tau\wedge\tau^*<t} &= \tau \wedge \tau^* \left(\mathbb{1}_{\tau<t} + \mathbb{1}_{\tau^*<t} - \mathbb{1}_{\tau<t,\tau^*<t}\right) \\ &= \tau_t \wedge \tau^* \mathbb{1}_{\tau<t} + \tau \wedge \tau_t^* \mathbb{1}_{\tau^*<t} - \tau_t \wedge \tau_t^* \mathbb{1}_{\tau<t,\tau^*<t} \end{aligned} \tag{7.6.1}$$

We focus on the first part of the sum (7.6.1). Additionally, we use the fact that τ_t can be chosen such that $\tau_t \leq t$ ([Jeulin p.73]), then

$$\begin{aligned} \tau_t \wedge \tau^* \mathbb{1}_{\tau<t} &= \tau_t \mathbb{1}_{\tau<t,\tau_t\leq\tau^*} + \tau^* \mathbb{1}_{\tau<t,\tau^*<\tau_t,\tau_t\leq t} \\ &= \tau_t \mathbb{1}_{\tau<t,\tau_t\leq\tau^*} + \tau_t^* \mathbb{1}_{\tau<t,\tau^*<\tau_t} \\ &= \tau_t \wedge \tau_t^* \mathbb{1}_{\tau<t}. \end{aligned}$$

As τ_t^* can also be chosen such that $\tau_t^* \leq t$, second part of (7.6.1) can be transformed in the similar way. Summing up above results we have

$$\begin{aligned} \tau \wedge \tau^* \mathbb{1}_{\tau\wedge\tau^*<t} &= \tau_t \wedge \tau_t^* \mathbb{1}_{\tau<t} + \tau_t \wedge \tau_t^* \mathbb{1}_{\tau^*<t} - \tau_t \wedge \tau_t^* \mathbb{1}_{\tau<t,\tau^*<t} \\ &= \tau_t \wedge \tau_t^* \left(\mathbb{1}_{\tau<t} + \mathbb{1}_{\tau^*<t} - \mathbb{1}_{\tau<t,\tau^*<t}\right) \\ &= \tau_t \wedge \tau_t^* \mathbb{1}_{\tau\wedge\tau^*<t}, \end{aligned}$$

which shows that $\tau \wedge \tau^*$ is again an honest time. □

Corollary 7.6.18 *The hypothesis (H') is satisfied between $\mathbb{F}$ and the initial and progressive enlargement of $\mathbb{F}$ with both $\tau \wedge \tau^*$ and $\tau \vee \tau^*$.*

Proof: It is well know that hypothesis (H') holds for honest time. □

7.6.10 Multiplicative Decomposition

This section is a part of [99].

Definition 7.6.19 *An $\mathbb{F}$-local martingale N belongs to the class $(\mathcal{C}_0)$, if it is strictly positive, with no positive jumps, and $\lim_{t\to\infty} N_t = 0$.*

For N be a local martingale which belongs to the class $(\mathcal{C}_0)$, with $N_0 = x$, we set $S_t = \sup_{s\leq t} N_s$. We consider the last time where N reaches its maximum over $[0,\infty]$, i.e., the last time where N equal S:

$$g \;=\; \sup\{t \geq 0: \quad N_t = S_\infty\} = \sup\{t \geq 0: \quad S_t - N_t = 0\}. \tag{7.6.2}$$

Lemma 7.6.20 ***(Doob's maximal identity)***
For any $a > 0$, we have:

$$\mathbb{P}(S_\infty > a) = \left(\frac{x}{a}\right) \wedge 1. \tag{7.6.3}$$

In particular, $\dfrac{x}{S_\infty}$ is a uniform random variable on $(0,1)$.
For any $\mathbb{F}$-stopping time ϑ, denoting $S^\vartheta = \sup_{u\geq\vartheta} N_u$:

$$\mathbb{P}\left(S^\vartheta > a|\mathcal{F}_\vartheta\right) = \left(\frac{N_\vartheta}{a}\right) \wedge 1, \tag{7.6.4}$$

Hence $\dfrac{N_\vartheta}{S^\vartheta}$ is also a uniform random variable on $(0,1)$, independent of $\mathcal{F}_\vartheta$.

Proof: The first part was done in Exercise 1.5.3. The second part is an application of the first one for the martingale $(N_{\vartheta+t}, t \geq 0)$ and the filtration $(\mathcal{F}_{\vartheta+t}, t \geq 0)$. □

Without loss of generality, we restrict our attention to the case $x = 1$.

Proposition 7.6.21 *The supermartingale $G_t = \mathbb{P}(g > t \mid \mathcal{F}_t)$ admits the multiplicative decomposition $G_t = \frac{N_t}{S_t}$, $t \geq 0$.*

Proof: We have the following equalities

$$\begin{aligned}\{g > t\} &= \{\exists\, u > t: \; S_u = N_u\} = \{\exists\, u > t: \; S_t \leq N_u\}\\ &= \left\{\sup_{u\geq t} N_u \geq S_t\right\} = \{S^t \geq S_t\}.\end{aligned}$$

Hence, from (7.6.4), we get: $\mathbb{P}(g > t \mid \mathcal{F}_t) = \frac{N_t}{S_t}$. □
From results given in Section 5.5.2

Corollary 7.6.22 *Any $\mathbb{F}$-local martingale X is a $\mathbb{F}^g$ semi martingale X with decomposition*

$$X_t = \widetilde{X}_t + \int_0^t \mathbb{1}_{\{g>s\}} \frac{d\langle X, N\rangle_s}{N_s} - \int_0^t \mathbb{1}_{\{g\leq s\}} \frac{d\langle X, N\rangle_s}{S_\infty - N_s},$$

where $\widetilde{X}$ is an $\mathbb{F}^g$-local martingale.

Proof: Let X be an $\mathbb{F}$-martingale which is in $\mathbb{H}^1$; the general case follows by localization.

$$X_t = \widetilde{X}_t + \int_0^t \mathbb{1}_{\{g>s\}} \frac{d\langle X, N\rangle_s}{N_s} - \int_0^t \mathbb{1}_{\{g\leq s\}} \frac{d\langle X, N\rangle_s}{S_\infty - N_s},$$

where $\widetilde{X}$ denotes an $\mathbb{F}^{(S_\infty)}$ martingale. Thus, $\widetilde{X}$, which is equal to:

$$X_t - \left(\int_0^t \mathbb{1}_{\{g>s\}} \frac{d\langle X,N\rangle_s}{N_s} - \int_0^t \mathbb{1}_{\{g\le s\}} \frac{d\langle X,N\rangle_s}{S_\infty - N_s}, \right),$$

is $\mathbb{F}^g$ adapted (recall that $\mathcal{F}_t^g \subset \mathcal{F}_t^{(S_\infty)}$), and hence it is an $\mathbb{F}^g$-martingale.

These results extend to honest times:

Theorem 7.6.23 *Let τ be an honest time. Then, under the conditions **(CA)**, the supermartingale $Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ admits the following additive and multiplicative representations: there exists a continuous and nonnegative local martingale N, with $N_0 = 1$ and $\lim_{t\to\infty} N_t = 0$, such that:*

$$\begin{aligned} Z_t &= \mathbb{P}(\tau > t|\mathcal{F}_t) = \frac{N_t}{S_t} \\ Z_t &= \mu_t - A_t. \end{aligned}$$

where these two representations are related as follows:

$$\begin{aligned} N_t &= \exp\left(\int_0^t \frac{d\mu_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d<\mu>_s}{Z_s^2} \right), \quad S_t = \exp(A_t); \\ \mu_t &= 1 + \int_0^t \frac{dN_s}{S_s} = \mathbb{E}(\log S_\infty \mid \mathcal{F}_t), \quad A_t = \log S_t. \end{aligned}$$

It follows that the intensity of τ is $d\Lambda = \frac{dS}{N}$, and since dS charges only $\{S_t = N_t\}$, one has $d\Lambda = \frac{dS}{S}$.

Exercise 7.6.24 Prove that $\exp(\lambda B_t - \frac{\lambda^2}{2}t)$ belongs to $(\mathcal{C}_0)$. ◁

✓CARACTERISATION of $\mathbb{G}$ martingales in terms of $\mathbb{F}$ martingales (Barlow)

7.7 Survival Process

Proposition 7.7.1 *Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a given filtered probability space. Assume that N is a $(\mathbb{P},\mathbb{F})$-local martingale and Λ an $\mathbb{F}$-adapted increasing process such that $0 < N_t e^{-\Lambda_t} < 1$ for $t > 0$ and $N_0 = 1 = e^{-\Lambda_0}$. Then, there exists (on an extended probability space) a probability $\mathbb{Q}$ which satisfies $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$) and a random time τ such that $\mathbb{Q}(\tau > t|\mathcal{F}_t) = N_t e^{-\Lambda_t}$.*

Proof: We give the proof when $\mathbb{F}$ is a Brownian filtration and $\Lambda_t = \int_0^t \lambda_u du$. For the general case, see [70, 71]. We shall construct, using a change of probability, a conditional probability $G_t(u)$ which admits the given survival process (i.e. $G_t(t) = G_t = N_t e^{-\Lambda_t}$). From the conditional probability, one can deduce a density process, hence one can construct a random time admitting $G_t(u)$ as conditional probability. We are looking for conditional probabilities with a particular form (the idea is linked with the results obtained in Subsection 6.2.5). Let us start with a model in which $\mathbb{P}(\tau > t|\mathcal{F}_t) = e^{-\Lambda_t}$, where $\Lambda_t = \int_0^t \lambda_s ds$ and let N be an $\mathbb{F}$-local martingale such that $0 \le N_t e^{-\Lambda_t} \le 1$.
The goal is to prove that there exists a $\mathbb{G}$-martingale L such that, setting $d\mathbb{Q} = Ld\mathbb{P}$
(i) $\mathbb{Q}|_{\mathcal{F}_\infty} = \mathbb{P}|_{\mathcal{F}_\infty}$
(ii) $\mathbb{Q}(\tau > t|\mathcal{F}_t) = N_t e^{-\Lambda_t}$
The $\mathbb{G}$-adapted process L

$$L_t = \ell_t \mathbb{1}_{t<\tau} + \ell_t(\tau)\mathbb{1}_{\tau \le t}$$

is a martingale if for any u, $(\ell_t(u), t \geq u)$ is a martingale and if $\mathbb{E}(L_t|\mathcal{F}_t)$ is a $\mathbb{F}$-martingale. Then, (i) is satisfied if

$$1 = \mathbb{E}(L_t|\mathcal{F}_t) = \ell_t e^{-\Lambda_t} + \int_0^t \ell_t(u)\lambda_u e^{-\Lambda_u} du$$

and (ii) implies that $\ell = N$ and $\ell_t(t) = \ell_t$. We are now reduced to find a family of martingales $\ell_t(u), t \geq u$ such that

$$\ell_u(u) = N_u,\ 1 = N_t e^{-\Lambda_t} + \int_0^t \ell_t(u)\lambda_u e^{-\Lambda_u} du$$

We restrict our attention to families ℓ of the form

$$\ell_t(u) = X_t Y_u,\ t \geq u$$

where X is a martingale such that

$$X_t Y_t = N_t,\ 1 = N_t e^{-\Lambda_t} + X_t \int_0^t Y_u \lambda_u e^{-\Lambda_u} du$$

. It is easy to show that

$$Y_t = Y_0 + \int_0^t e^{\Lambda_u} d(\frac{1}{X_u})$$

In a Brownian filtration case, there exists a process ν such that $dN_t = \nu_t N_t dW_t$ and the positive martingale X is of the form $dX_t = x_t X_t dW_t$. Then, using the fact that integration by parts implies

$$d(X_t Y_t) = Y_t dX_t - e^{\Lambda_t} \frac{1}{X_t} dX_t = x_t(X_t Y_t - e^{\Lambda_t}) dW_t = dN_t\,,$$

we are lead to chose

$$x_t = \frac{\nu_t G_t}{G_t - 1}$$

Chapter 8

Initial and Progressive Enlargements with $(\mathcal{E})$-times

We consider a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual hypotheses of right-continuity and completeness, and where $\mathcal{F}_0$ is the trivial σ-field. Let τ be a finite random time (i.e., a finite non-negative random variable) with law ν, $\nu(du) = \mathbb{P}(\tau \in du)$. We assume that ν has no atoms and has $\mathbb{R}_+$ as support.

We denote by $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{O}(\mathbb{F})$) the predictable (resp. optional) σ-algebra corresponding to $\mathbb{F}$ on $\mathbb{R}^+ \times \Omega$. We consider the three nested filtrations

$$\mathbb{F} \subset \mathbb{G} \subset \mathbb{F}^{(\tau)}$$

where $\mathbb{G}$ and $\mathbb{F}^{(\tau)}$ stand, respectively, for the *progressive* and the *initial* enlargement of $\mathbb{F}$ with the random time τ.

In this chapter, our standing assumption is the following:

Hypothesis 8.0.2 $(\mathcal{E})$***-Hypothesis***
The $\mathbb{F}$*-(regular) conditional law of* τ *is equivalent to the law of* τ*. Namely,*

$$\mathbb{P}(\tau \in du|\mathcal{F}_t) \sim \nu(du) \quad \textit{for every } t \geq 0, \ \ \mathbb{P} - a.s.$$

We shall call $(\mathcal{E})$-times random times which satisfy $(\mathcal{E})$-Hypothesis. This assumption, in the case when $t \in [0, T]$, corresponds to the *equivalence assumption* in Föllmer and Imkeller [52] and in Amendinger's thesis [5, Assumption 0.2] and to hypothesis (HJ) in the papers by Grorud and Pontier (see, e.g., [57]). Under the $(\mathcal{E})$-Hypothesis, we address the following problems:

- Characterization of $\mathbb{G}$-martingales and $\mathbb{F}^{(\tau)}$-martingales in terms of $\mathbb{F}$-martingales;
- Canonical decomposition of an $\mathbb{F}$-martingale, as a semimartingale, in $\mathbb{G}$ and $\mathbb{F}^{(\tau)}$;
- Predictable Representation Theorem in $\mathbb{G}$ and $\mathbb{F}^{(\tau)}$.

This chapter is based on [26] and [5].

8.1 Preliminaries

The exploited idea is the following: assuming that the $\mathbb{F}$-conditional law of τ is equivalent to the law of τ, after an *ad hoc* change of probability measure, the problem reduces to the case where τ

and $\mathbb{F}$ are independent. Under this newly introduced probability measure, working in the initially enlarged filtration is "easy". Then, under the original probability measure, for the initially enlarged filtration, the results are achieved by means of Girsanov's theorem. Finally, by projection, one obtains the results of interest in the progressively enlarged filtration (notice that, alternatively, they can be obtained with another application of Girsanov's theorem, starting from the newly introduced probability measure, with respect to the progressively enlarged filtration).

The "change of probability measure" viewpoint for treating problems on enlargement of filtrations was remarked in the early 80's and developed by Song in [105] (see also Jacod [65, Section 5]). This is also the point of view adopted by Gasbarra et al. in [56] while applying the Bayesian approach to study the impact of the initial enlargement of filtration on the characteristic triplet of a semimartingale. For what concerns the idea of recovering the results in the progressively enlarged filtration starting from the ones in the initially enlarged one, we have to cite Yor [113].

Amongst the consequences of the ($\mathcal{E}$)-Hypothesis, one has the existence and regularity of the conditional density, for which we refer to Amendiger's reformulation (see [5, Remarks, p. 17]) of Jacod's result [65, (Lemma 1.8]): there exists a strictly positive $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function $(t, \omega, u) \to p_t(\omega, u)$, such that for every $u \in \mathbb{R}^+$, $p(u)$ is a càdlàg $(\mathbb{P}, \mathbb{F})$-martingale and

$$\mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty p_t(u) \nu(du) \quad \text{for every } t \geq 0, \;\; \mathbb{P} - a.s.$$

In particular, $p_0(u) = 1$ for every $u \in \mathbb{R}^+$ and $\int_0^\infty p_t(u)\nu(du) = 1, \forall t$. This family of processes p is called the $(\mathbb{P}, \mathbb{F})$-*conditional density* of τ with respect to ν, or the *density* of τ if there is no ambiguity.

Furthermore, under the ($\mathcal{E}$)-Hypothesis, the assumption that ν has no atoms implies that the default time τ avoids the $\mathbb{F}$-stopping times, i.e., $\mathbb{P}(\tau = \xi) = 0$ for every $\mathbb{F}$-stopping time ξ (see, e.g., El Karoui et al. [42, Corollary 2.2]).

✓GIVE DETAILS

It was shown in [5, Proposition 1.10] that the strict positiveness of p implies the right-continuity of the filtration $\mathbb{F}^{(\tau)}$.

In the sequel, we will consider the right-continuous version of all the martingales.

Now, we consider the change of probability measure introduced, independently, by Grorud and Pontier in [57] and by Amendinger in [5] and we recall the result established in Lemma 5.5.1.

Lemma 8.1.1 *Let L be the $(\mathbb{P}, \mathbb{F}^{(\tau)})$-martingale defined as $L_t = \frac{1}{p_t(\tau)}$, $t \geq 0$, and $\mathbb{P}^*$ the probability measure defined on $\mathbb{F}^{(\tau)}$ as*

$$d\mathbb{P}^*{}_{|\mathcal{F}_t^{(\tau)}} = L_t \, d\mathbb{P}_{|\mathcal{F}_t^{(\tau)}} = \frac{1}{p_t(\tau)} d\mathbb{P}_{|\mathcal{F}_t^{(\tau)}} .$$

Under $\mathbb{P}^*$, the random time τ is independent of $\mathcal{F}_t$ for any $t \geq 0$ and, moreover

$$\mathbb{P}^*{}_{|\mathcal{F}_t} = \mathbb{P}_{|\mathcal{F}_t} \; \textit{for any } t \geq 0, \qquad \mathbb{P}^*{}_{|\sigma(\tau)} = \mathbb{P}_{|\sigma(\tau)}.$$

The above properties imply that $\mathbb{P}^*(\tau \in du | \mathcal{F}_t) = \mathbb{P}^*(\tau \in du)$, so that the $(\mathbb{P}^*, \mathbb{F})$-density of τ, denoted by $p^*(u), u \geq 0$, is a constant equal to one, $\mathbb{P}^* \otimes \nu$-a.s.

Remark 8.1.2 The probability measure $\mathbb{P}^*$, being defined on $\mathcal{F}_t$ for $t \geq 0$, is (uniquely) defined on $\mathcal{F}_\infty = \bigvee_{t \geq 0} \mathcal{F}_t$. Then, as τ is independent of $\mathbb{F}$ under $\mathbb{P}^*$, it immediately follows that τ is also independent of $\mathcal{F}_\infty$, under $\mathbb{P}^*$. However, one can not claim that: "$\mathbb{P}^*$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_\infty^{(\tau)}$", since we do not know *a priori* whether $\frac{1}{p(\tau)}$ is a *closed* $(\mathbb{P}, \mathbb{F}^{(\tau)})$-martingale or not. A similar problem is studied by Föllmer and Imkeller in [52] (it is therein called "paradox") in the case where the reference (canonical) filtration is enlarged by means of the information about the endpoint at time $t = 1$. In our setting, it corresponds to the case where $\tau \in \mathcal{F}_\infty$ and $\tau \notin \mathcal{F}_t, \forall\, t$. In the Brownian bridge case, the conditional law of B_1 w.r.t. $\mathcal{F}_t$ is the Dirac measure for $t = 1$.

Notation 8.1.3 *In this chapter, as we mentioned, we deal with three different levels of information and two equivalent probability measures. In order to distinguish objects defined under $\mathbb{P}$ and under $\mathbb{P}^*$, we will use, in this chapter, a superscript $*$ when working under $\mathbb{P}^*$. For example, $\mathbb{E}$ and $\mathbb{E}^*$ stand for the expectations under $\mathbb{P}$ and $\mathbb{P}^*$, respectively. For what concerns the filtrations, when necessary, we will use the following illustrating notation: $x, X, X^{(\tau)}$ to denote processes adapted to $\mathbb{F}, \mathbb{G}$ and $\mathbb{F}^{(\tau)}$, respectively.*

Let $x = (x_t, t \geq 0)$ be a $(\mathbb{P}, \mathbb{F})$-martingale. Since $\mathbb{P}$ and $\mathbb{P}^*$ coincide on $\mathbb{F}$, x is a $(\mathbb{P}^*, \mathbb{F})$-martingale, hence, using the fact that τ is independent of $\mathbb{F}$ under $\mathbb{P}^*$, a $(\mathbb{P}^*, \mathbb{G})$-martingale (and also a $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$-martingale). Because of these facts, the measure $\mathbb{P}^*$ is called by Amendinger *"martingale preserving probability measure under initial enlargement of filtrations"*.

8.1.1 Expectation and projection tools

Lemma 8.1.4 *Let $Y_t^{(\tau)} = y_t(\tau)$ be an $\mathcal{F}_t^{(\tau)}$-measurable random variable.*
(i) If $y_t(\tau)$ is $\mathbb{P}$-integrable and $y_t(\tau) = 0$ $\mathbb{P}$-a.s. then, for ν-a.e. $u \geq 0$, $y_t(u) = 0$ $\mathbb{P}$-a.s.
(ii) For $s \leq t$ one has, $\mathbb{P}$-a.s. (or, equivalently, $\mathbb{P}^*$-a.s.):
if $y_t(\tau)$ is $\mathbb{P}^*$-integrable and if $y_t(u)$ is $\mathbb{P}$ (or $\mathbb{P}^*$)-integrable for any $u \geq 0$,

$$\mathbb{E}^*\big(y_t(\tau)|\mathcal{F}_s^{(\tau)}\big) = \mathbb{E}^*\big(y_t(u)|\mathcal{F}_s\big)\Big|_{u=\tau} = \mathbb{E}\big(y_t(u)|\mathcal{F}_s\big)\Big|_{u=\tau}; \tag{8.1.1}$$

if $y_t(\tau)$ is $\mathbb{P}$-integrable

$$\mathbb{E}\big(y_t(\tau)|\mathcal{F}_s^{(\tau)}\big) = \frac{1}{p_s(\tau)}\mathbb{E}\big(y_t(u)p_t(u)|\mathcal{F}_s\big)\Big|_{u=\tau}. \tag{8.1.2}$$

Proof: (i) We have, by applying Fubini-Tonelli's Theorem,

$$0 \quad = \quad \mathbb{E}\big(|y_t(\tau)|\big) = \mathbb{E}\Big(\mathbb{E}\big(|y_t(\tau)|\big|\mathcal{F}_t\big)\Big) = \mathbb{E}\Big(\int_0^\infty |y_t(u)|p_t(u)\nu(du)\Big).$$

Then $\int_0^\infty |y_t(u)|p_t(u)\nu(du) = 0$ $\mathbb{P}$-a.s. and, given that $p_t(u)$ is strictly positive for ν almost every u, we have that, for ν-almost every u, $y_t(\cdot, u) = 0$ $\mathbb{P}$-a.s.
(ii) The first equality in (8.1.1) is straightforward for elementary random variables of the form $f(\tau)x_t$, given the independence between τ and $\mathcal{F}_t$, for any $t \geq 0$. It is extended to $\mathcal{F}_t^{(\tau)}$-measurable r.v's via the monotone class theorem. The second equality follows from the fact that $\mathbb{P}$ and $\mathbb{P}^*$ coincide on $\mathcal{F}_t$, for any $t \geq 0$.

The result (8.1.2) is an immediate consequence of (8.1.1), since it suffices, by means of (conditional) Bayes' formula, to pass under the measure $\mathbb{P}^*$. More precisely, for $s < t$, we have

$$\mathbb{E}\big(y_t(\tau)|\mathcal{F}_s^{(\tau)}\big) = \frac{\mathbb{E}^*\big(y_t(\tau)p_t(\tau)|\mathcal{F}_s^{(\tau)}\big)}{\mathbb{E}^*\big(p_t(\tau)|\mathcal{F}_s^{(\tau)}\big)} = \frac{1}{p_s(\tau)}\mathbb{E}\big(y_t(u)p_t(u)|\mathcal{F}_s\big)_{|u=\tau},$$

where in the last equality we have used the previous result (8.1.1) and the fact that $p(\tau)$ is a $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$-martingale. Note that if $y_t(\tau)$ is $\mathbb{P}$-integrable, then $\mathbb{E}(\int_0^\infty |y_t(u)|p_t(u)\nu(du)) = \mathbb{E}(|y_t(\tau)|) < \infty$, which implies that $\mathbb{E}(|y_t(u)|p_t(u)) < \infty$. □

The Azéma supermartingale associated with τ under the probability measure $\mathbb{P}$ (resp. $\mathbb{P}^*$) is

$$G_t \quad := \quad \mathbb{P}(\tau > t|\mathcal{F}_t) = \int_t^\infty p_t(u)\nu(du), \tag{8.1.3}$$

$$G^*(t) \quad := \quad \mathbb{P}^*(\tau > t|\mathcal{F}_t) = \mathbb{P}^*(\tau > t) = \mathbb{P}(\tau > t) = \int_t^\infty \nu(du) = G(t). \tag{8.1.4}$$

Note, in particular, that G is a $(\mathbb{P}, \mathbb{F})$ super-martingale, whereas $G^*(\cdot)$ is a (deterministic) continuous and decreasing function. Furthermore, it is clear that, under the ($\mathcal{E}$)-Hypothesis and the hypothesis that the support of ν is $\mathbb{R}_+$, G and G^* do not vanish.

✓details

Lemma 8.1.5 *Let $Y_t^{(\tau)} = y_t(\tau)$ be an $\mathcal{F}_t^{(\tau)}$-measurable, $\mathbb{P}$-integrable random variable. Then, for $s \leq t$,*

$$\mathbb{E}(Y_t^{(\tau)}|\mathcal{G}_s) = \mathbb{E}(y_t(\tau)|\mathcal{G}_s) = \tilde{y}_s 1\!\!1_{s<\tau} + \hat{y}_s(\tau) 1\!\!1_{\tau \leq s},$$

with

$$\begin{aligned} \tilde{y}_s &= \frac{1}{G_s}\mathbb{E}\Big(\int_s^{+\infty} y_t(u)p_t(u)\nu(du)|\mathcal{F}_s\Big) , \\ \hat{y}_s(u) &= \frac{1}{p_s(u)}\mathbb{E}\big(y_t(u)p_t(u)|\mathcal{F}_s\big) . \end{aligned}$$

Proof: From the above Proposition 7.2.1, it is clear that $\mathbb{E}(y_t(\tau)|\mathcal{G}_s)$ can be written in the form $\tilde{y}_s 1\!\!1_{s<\tau} + \hat{y}_s(\tau)1\!\!1_{\tau\leq s}$. On the set $\{s < \tau\}$, we have, applying the key Lemma and using the ($\mathcal{E}$)-Hypothesis (see also [42] for analogous computations),

$$\begin{aligned} 1\!\!1_{s<\tau}\mathbb{E}(y_t(\tau)|\mathcal{G}_s) &= 1\!\!1_{s<\tau}\frac{\mathbb{E}\left[\mathbb{E}(y_t(\tau)1\!\!1_{s<\tau}|\mathcal{F}_t)|\mathcal{F}_s\right]}{G_s} \\ &= 1\!\!1_{s<\tau}\frac{1}{G_s}\mathbb{E}\left(\int_s^{+\infty} y_t(u)p_t(u)\nu(du)|\mathcal{F}_s\right) =: 1\!\!1_{s<\tau}\tilde{y}_s. \end{aligned}$$

On the complementary set, we have, by applying Lemma 8.1.4,

$$1\!\!1_{\tau\leq s}\mathbb{E}(y_t(\tau)|\mathcal{G}_s) = 1\!\!1_{\tau\leq s}\mathbb{E}\left[\mathbb{E}(y_t(\tau)|\mathcal{G}_s^\tau)|\mathcal{G}_s\right] = 1\!\!1_{\tau\leq s}\frac{1}{p_s(\tau)}\mathbb{E}\big(y_t(u)p_t(u)|\mathcal{F}_s\big)\Big|_{u=\tau} =: 1\!\!1_{\tau\leq s}\hat{y}_s(\tau).$$

□

For $s > t$, we obtain $\mathbb{E}(Y_t^{(\tau)}|\mathcal{G}_s) = \frac{1}{G_s}\int_s^\infty y_t(u)p_s(u)\nu(du)1\!\!1_{s<\tau} + y_t(\tau)1\!\!1_{\tau\leq s}$.

As an application, projecting the martingale L (defined earlier as $L_t = \frac{1}{p_t(\tau)}, t \geq 0$) on $\mathbb{G}$ yields to the corresponding Radon-Nikodým density on $\mathbb{G}$:

$$d\mathbb{P}^*{}_{|\mathcal{G}_t} = \ell_t \; d\mathbb{P}_{|\mathcal{G}_t} \; ,$$

with

$$\begin{aligned} \ell_t &:= \mathbb{E}(L_t|\mathcal{G}_t) = 1\!\!1_{t<\tau}\frac{1}{G_t}\int_t^\infty \nu(du) + 1\!\!1_{\tau\leq t}\frac{1}{p_t(\tau)} \\ &= 1\!\!1_{t<\tau}\frac{G(t)}{G_t} + 1\!\!1_{\tau\leq t}\frac{1}{p_t(\tau)} \; . \end{aligned}$$

Proposition 8.1.6 *The Azéma super-martingale G, introduced in Equation (8.1.3), admits the Doob-Meyer decomposition $G_t = \mu_t - \int_0^t p_u(u)\nu(du), t \geq 0$, where μ is the $\mathbb{F}$-martingale defined as*

$$\mu_t := 1 - \int_0^t \left(p_t(u) - p_u(u)\right)\nu(du)$$

The intensity of τ is $\lambda_t = \frac{p_t(t)}{G_t}$.

PROOF: From the definition of G and using the fact that $p(u)$ is martingale,

$$G_t + \int_0^t p_u(u)\nu(du) = \int_t^\infty p_t(u)\nu(du) + \int_0^t p_u(u)\nu(du) = \mathbb{E}(\int_0^\infty p_u(u)\nu(du)|\mathcal{F}_t)$$

□

We now recall some useful facts concerning the compensated martingale of H. We know, from the general theory (see, for example, [42]), that denoting by H the default indicator process $H_t = 1\!\!1_{\tau\le t}, t \ge 0$, the process M defined as

$$M_t := H_t - \int_0^{t\wedge\tau} \lambda_s\ \nu(ds), \quad t \ge 0, \tag{8.1.5}$$

with $\lambda_t = \frac{p_t(t)}{G_t}$, is a $(\mathbb{P}, \mathbb{G})$-martingale and that

$$M_t^* := H_t - \int_0^{t\wedge\tau} \lambda^*(s)\ \nu(ds), \quad t \ge 0, \tag{8.1.6}$$

with $\lambda^*(t) = \frac{1}{G(t)}$, is a $(\mathbb{P}^*, \mathbb{G})$-martingale. Furthermore, since λ^* is deterministic, M^* (being $\mathbb{H}$-adapted) is a $(\mathbb{P}^*, \mathbb{H})$-martingale, too.

We conclude this subsection with the following two propositions, concerning the predictable projection, respectively on $\mathbb{F}$ and on $\mathbb{G}$, of a $\mathbb{F}^{(\tau)}$-predictable process. The first result is due to Jacod [65, Lemme 1.10].

Proposition 8.1.7 *Let $Y^{(\tau)} = y(\tau)$ be a $\mathbb{F}^{(\tau)}$-predictable, positive or bounded, process. Then, the $\mathbb{P}$-predictable projection of $Y^{(\tau)}$ on $\mathbb{F}$ is given by*

$$^{(p,\mathbb{F})}(Y^\tau)_t = \int_0^\infty y_t(u)p_{t-}(u)\nu(du)\ .$$

PROOF: It is obtained by a monotone class argument and by using the definition of density of τ, writing, for "elementary" processes, $Y_t^{(\tau)} := y_t f(\tau)$, with y a bounded $\mathbb{F}$-predictable process and f a bounded Borel function. For this, we refer to the proof in Jacod [65, Lemme 1.10]. □

Proposition 8.1.8 *Let $Y^{(\tau)} = y(\tau)$ be a $\mathbb{F}^{(\tau)}$-predictable, positive or bounded, process. Then, the $\mathbb{P}$-predictable projection of $Y^{(\tau)}$ on $\mathbb{G}$ is given by*

$$^{(p,\mathbb{G})}(Y^{(\tau)})_t = 1\!\!1_{t\le\tau}\frac{1}{G_{t-}}\int_t^\infty y_t(u)p_{t-}(u)\nu(du) + 1\!\!1_{\tau<t}y_t(\tau)\ .$$

PROOF: By the definition of predictable projection, we know (from Proposition 7.2.1 (ii)) that we are looking for a (unique) process of the form

$$^{(p\,\mathbb{G})}(Y^\tau)_t = \tilde{y}_t 1\!\!1_{t\le\tau} + \hat{y}_t(\tau)1\!\!1_{\tau<t}, \quad t \ge 0,$$

where $\tilde{y}$ is $\mathbb{F}$-predictable, positive or bounded, and $(t, \omega, u) \mapsto \hat{y}_t(\omega, u)$ is a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable positive or bounded function, to be identified.

- On the predictable set $\{\tau < t\}$, being $Y^{(\tau)}$ a $\mathbb{F}^{(\tau)}$-predictable, positive or bounded, process (recall Proposition 5.1.1 (ii)), we immediately find $\hat{y}(\tau) = y(\tau)$;
- On the complementary set $\{t \le \tau\}$, introducing the $\mathbb{G}$-predictable process

$$Y := {}^{(p\,\mathbb{G})}(Y^{(\tau)})$$

it is possible to use Jeulin [73, Remark 4.5, page 64] (see also Dellacherie et al. [35, Ch. XX, page 186]), to write

$$Y 1\!\!1_{]\!]0,\tau]\!]} = \frac{1}{G_-}\ {}^{(p\,\mathbb{F})}\big(Y 1\!\!1_{]\!]0,\tau]\!]}\big) 1\!\!1_{]\!]0,\tau]\!]} = \frac{1}{G_-}\ {}^{(p\,\mathbb{F})}\Big({}^{(p\,\mathbb{G})}(Y^{(\tau)}) 1\!\!1_{]\!]0,\tau]\!]}\Big) 1\!\!1_{]\!]0,\tau]\!]}.$$

We then have, being $1\!\!1_{]\!]0,\tau]\!]}$, by definition, $\mathbb{G}$-predictable (recall that τ is a $\mathbb{G}$-stopping time),

$$Y 1\!\!1_{]\!]0,\tau]\!]} = \frac{1}{G_-}\ {}^{(p\,\mathbb{F})}\Big(Y^{(\tau)} 1\!\!1_{]\!]0,\tau]\!]}\Big) 1\!\!1_{]\!]0,\tau]\!]},$$

where the last equality follows by the definition of predictable projection, being $\mathbb{F} \subset \mathbb{G}$. Finally, given the result in Proposition 8.1.7, we have

$${}^{(p\,\mathbb{F})}\Big(Y^{(\tau)} 1\!\!1_{]\!]0,\tau]\!]}\Big)_t = \int_t^{+\infty} y_t(u) p_{t-}(u) \nu(du)$$

and the proposition is proved.

□

8.2 Martingales' characterization

The aim of this section is to characterize $(\mathbb{P}, \mathbb{F}^{(\tau)})$ and $(\mathbb{P}, \mathbb{G})$-martingales in terms of $(\mathbb{P}, \mathbb{F})$-martingales.

Proposition 8.2.1 ***Characterization of*** $(\mathbb{P}, \mathbb{F}^{(\tau)})$***-martingales in terms of*** $(\mathbb{P}, \mathbb{F})$***-martingales***
A process $Y^{(\tau)} = y(\tau)$ *is a* $(\mathbb{P}, \mathbb{F}^{(\tau)})$*-martingale if and only if* $(y_t(u) p_t(u), t \geq 0)$ *is a* $(\mathbb{P}, \mathbb{F})$*-martingale, for* ν*-almost every* $u \geq 0$.

Proof: The sufficiency is a direct consequence of Proposition 5.1.1 and Lemma 8.1.4 (ii).
Conversely, assume that $y(\tau)$ is an $\mathbb{F}^{(\tau)}$-martingale. Then, for $s \leq t$, from Lemma 8.1.4 (ii),

$$y_s(\tau) = \mathbb{E}\left(y_t(\tau) | \mathcal{F}_s^{(\tau)}\right) = \frac{1}{p_s(\tau)} \mathbb{E}\big(y_t(u) p_t(u) | \mathcal{F}_s\big)_{|u=\tau}$$

and the result follows from Lemma 8.1.4 (i). □

Passing to the progressive enlargement setting, we state and prove a martingale characterization result, essentially established by El Karoui et al. in [42, Theorem 5.7].

Proposition 8.2.2 ***Characterization of*** $(\mathbb{P}, \mathbb{G})$ ***martingales in terms of*** $(\mathbb{P}, \mathbb{F})$***-martingales***
A $\mathbb{G}$*-adapted process* $Y_t := \tilde{y}_t 1\!\!1_{t<\tau} + \hat{y}_t(\tau) 1\!\!1_{\tau \leq t}, t \geq 0$, *is a* $(\mathbb{P}, \mathbb{G})$*-martingale if and only if the following two conditions are satisfied*

(i) for ν*-a.e* u, $\big(\hat{y}_t(u) p_t(u), t \geq u\big)$ *is a* $(\mathbb{P}, \mathbb{F})$*-martingale;*

(ii) the process $m = (m_t, t \geq 0)$*, given by*

$$m_t := \mathbb{E}(Y_t | \mathcal{F}_t) = \tilde{y}_t G_t + \int_0^t \hat{y}_t(u) p_t(u) \nu(du)\ , \tag{8.2.1}$$

is a $(\mathbb{P}, \mathbb{F})$*-martingale.*

PROOF: For the necessity, in a first step, we show that we can reduce our attention to the case where Y is u.i.: indeed, let Y be a $(\mathbb{P}, \mathbb{G})$-martingale. For any T, let $Y^T = (Y_{t\wedge T}, t \geq 0)$ be the associated stopped martingale, which is u.i. Assuming that the result is established for u.i. martingales will prove that the processes in *(i)* and *(ii)* are martingales up to time T. Since T can be chosen as large as possible, we shall have the result.

Assume, then, that Y is a u.i. $(\mathbb{P}, \mathbb{G})$-martingale. From Proposition 1.2.3, $Y_t = \mathbb{E}(Y_t^{(\tau)}|\mathcal{G}_t)$ for some $(\mathbb{P}, \mathbb{F}^{(\tau)})$-martingale $Y^{(\tau)}$. Proposition 8.2.1, then, implies that $Y_t^{(\tau)} = y_t(\tau)$, where for ν-almost every $u \geq 0$ the process $\big(y_t(u)p_t(u), t \geq 0\big)$ is a $(\mathbb{P}, \mathbb{F})$-martingale. One then has

$$\mathbb{1}_{\tau\leq t}\hat{y}_t(\tau) = \mathbb{1}_{\tau\leq t}Y_t = \mathbb{1}_{\tau\leq t}\mathbb{E}(Y_t^{(\tau)}|\mathcal{G}_t) = \mathbb{E}(\mathbb{1}_{\tau\leq t}Y_t^{(\tau)}|\mathcal{G}_t) = \mathbb{1}_{\tau\leq t}y_t(\tau) \ ,$$

which implies, in view of Lemma 8.1.4(i), that for ν-almost every $u \leq t$, the identity $y_t(u) = \hat{y}_t(u)$ holds $\mathbb{P}$-almost surely. So, *(i)* is proved. Moreover, Y being a $(\mathbb{P}, \mathbb{G})$-martingale, its projection on the smaller filtration $\mathbb{F}$, namely the process m in (8.2.1), is a $(\mathbb{P}, \mathbb{F})$-martingale.

Conversely, assuming *(i)* and *(ii)*, we verify that $\mathbb{E}(Y_t|\mathcal{G}_s) = Y_s$ for $s \leq t$. We start by noting that

$$\mathbb{E}(Y_t|\mathcal{G}_s) = \mathbb{1}_{s<\tau}\frac{1}{G_s}\mathbb{E}(Y_t\mathbb{1}_{s<\tau}|\mathcal{F}_s) + \mathbb{1}_{\tau\leq s}\mathbb{E}(Y_t\mathbb{1}_{\tau\leq s}|\mathcal{G}_s) \ . \tag{8.2.2}$$

We then compute the two conditional expectations in (8.2.2):

$$\begin{aligned}
\mathbb{E}(Y_t\mathbb{1}_{s<\tau}|\mathcal{F}_s) &= \mathbb{E}(Y_t|\mathcal{F}_s) - \mathbb{E}(Y_t\mathbb{1}_{\tau\leq s}|\mathcal{F}_s) \\
&= \mathbb{E}(m_t|\mathcal{F}_s) - \mathbb{E}\big(\mathbb{E}(\hat{y}_t(\tau)\mathbb{1}_{\tau\leq s}|\mathcal{F}_t)|\mathcal{F}_s\big) \\
&= m_s - \mathbb{E}\Big(\int_0^s \hat{y}_t(u)p_t(u)\nu(du)|\mathcal{F}_s\Big) \\
&= \tilde{y}_s G_s + \int_0^s \hat{y}_s(u)p_s(u)\nu(du) - \int_0^s \hat{y}_s(u)p_s(u)\nu(du) = \tilde{y}_s G_s \ ,
\end{aligned}$$

where we used Fubini's theorem and the condition *(i)* to obtain the next-to-last identity.
Also, an application of Lemma 8.1.5 yields to

$$\begin{aligned}
\mathbb{E}(Y_t\mathbb{1}_{\tau\leq s}|\mathcal{G}_s) &= \mathbb{E}(\hat{y}_t(\tau)\mathbb{1}_{\tau\leq s}|\mathcal{G}_s) = \mathbb{1}_{\tau\leq s}\frac{1}{p_s(\tau)}\mathbb{E}\big(\hat{y}_t(u)p_t(u)|\mathcal{F}_s\big)_{|u=\tau} \\
&= \mathbb{1}_{\tau\leq s}\frac{1}{p_s(\tau)}\hat{y}_s(\tau)p_s(\tau) = \mathbb{1}_{\tau\leq s}\hat{y}_s(\tau)
\end{aligned}$$

where the next-to-last identity holds in view of the condition *(ii)*. □

✓ADD NICOLE RESULTS

8.3 Canonical decomposition

In this section, we work under $\mathbb{P}$ and we show that any $\mathbb{F}$-local martingale x is a semi-martingale in both the initially enlarged filtration $\mathbb{F}^{(\tau)}$ and in the progressively enlarged filtration $\mathbb{G}$, and that any $\mathbb{G}$-martingale is a $\mathbb{F}^{(\tau)}$-semi-martingale. We also provide the canonical decomposition of any $\mathbb{F}$-local martingale as a semi-martingale in $\mathbb{F}^{(\tau)}$ and in $\mathbb{G}$. Under the assumption that the $\mathbb{F}$-conditional law of τ is absolutely continuous w.r.t. the law of τ, these questions were answered in Chapter 5, in the initial enlargement setting, and in [42] and [67], in the progressive enlargement case. Our aim here is to retrieve their results in an alternative manner.

We will need the following technical result, concerning the existence of the predictable bracket $\langle x, p_{.}(u)\rangle$. From [65, Theorem 2.5 a)], it follows immediately that, under the $(\mathcal{E})$-Hypothesis, for every $(\mathbb{P}, \mathbb{F})$-(local)martingale x, there exists a ν-negligible set B (depending on x), such that $\langle x, p_{.}(u)\rangle$ is well-defined for $u \notin B$. Hereafter, by $\langle x, p_{.}(\tau)\rangle_s$ we mean $\langle x, p_{.}(u)\rangle_s|_{u=\tau}$.

Furthermore, according to [65, Theorem 2.5 b)], under the $(\mathcal{E})$-Hypothesis, there exists an $\mathbb{F}$-predictable increasing process A and a $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$-measurable function $(t,\omega,u)\to k_t(\omega,u)$ such that, for any $u\notin B$ and for all $t\geq 0$,

$$\langle x, p_.(u)\rangle_t = \int_0^t k_s(u)p_{s-}(u)dA_s \quad \text{a.s.} \tag{8.3.1}$$

(the two processes A and k depend on x, however, to keep simple notation, we do not write $A^{(x)}$, nor $k^{(x)}$).

Moreover,

$$\int_0^t |k_s(\tau)|dA_s \;<\; \infty \quad \text{a.s., for any } t>0. \tag{8.3.2}$$

The following two propositions provide, under the $(\mathcal{E})$-Hypothesis, the canonical decomposition of any $(\mathbb{P},\mathbb{F})$- local martingale x in the enlarged filtrations $\mathbb{F}^{(\tau)}$ and $\mathbb{G}$, respectively. The first result is due to Jacod [65, Theorem 2.5 c)]). Our proof is easier (mainly because we do not prove the difficult regularity results obtained by Jacod), but less general. The interest is that we show the power of the change of probability methodology.

Proposition 8.3.1 ***Canonical Decomposition in*** $\mathbb{F}^{(\tau)}$
Any $(\mathbb{P},\mathbb{F})$-local martingale x is a $(\mathbb{P},\mathbb{F}^{(\tau)})$-semimartingale with canonical decomposition

$$x_t = X_t^{(\tau)} + \int_0^t \frac{d\langle x, p_.(\tau)\rangle_s}{p_{s-}(\tau)},$$

where $X^{(\tau)}$ is a $(\mathbb{P},\mathbb{F}^{(\tau)})$-local martingale.

Proof: If x is a $(\mathbb{P},\mathbb{F})$-martingale, it is a $(\mathbb{P}^*,\mathbb{F}^{(\tau)})$-martingale, too (Indeed, since $\mathbb{P}$ and $\mathbb{P}^*$ are equal on $\mathbb{F}$, x is a $(\mathbb{P}^*,\mathbb{F})$ martingale, hence, using the fact that τ is $\mathbb{P}^*$ independe=ant of $\mathbb{F}$, it is a $(\mathbb{P}^*,\mathbb{G})$ martingale). Noting that $d\mathbb{P}=p_t(\tau)d\mathbb{P}^*$ on $\mathcal{G}_t^\tau$, Girsanov's theorem tells us that the process $X^{(\tau)}$, defined by

$$X_t^{(\tau)} = x_t - \int_0^t \frac{d\langle x, p_.(\tau)\rangle_s}{p_{s-}(\tau)}$$

is a $(\mathbb{P},\mathbb{F}^{(\tau)})$-martingale. □

Now, any $(\mathbb{P},\mathbb{F})$-local martingale is a $\mathbb{G}$-adapted process and a $(\mathbb{P},\mathbb{F}^{(\tau)})$ semi-martingale (from the above Proposition 8.3.1), so in view of Stricker's theorem in [108], it is also a $\mathbb{G}$ semi-martingale. The following proposition aims to obtain the $\mathbb{G}$-canonical decomposition of an $\mathbb{F}$-local martingale. We refer to [67] for an alternative proof.

The following lemma provides a formula for the predictable quadratic covariation process $\langle x, G\rangle = \langle x,\mu\rangle$ in terms of the density p.

Lemma 8.3.2 *Let x be a $(\mathbb{P},\mathbb{F})$-local martingale and μ the $\mathbb{F}$-martingale part in the Doob-Meyer decomposition of G. If kp_- is $dA\otimes d\nu$-integrable, then*

$$\langle x,\mu\rangle_t = \int_0^t dA_s \int_s^\infty \nu(du)k_s(u)p_{s-}(u), \tag{8.3.3}$$

where k was introduced in Equation (8.3.1).

PROOF: First consider the right-hand-side of (8.3.3), that is, by definition, predictable, and apply Fubini's Theorem

$$\begin{aligned}
\xi_t &:= \int_0^t dA_s \int_s^\infty k_s(u)p_{s-}(u)\nu(du) \\
&= \int_0^t dA_s \int_s^t k_s(u)p_{s-}(u)\nu(du) + \int_0^t dA_s \int_t^\infty k_s(u)p_{s-}(u)\nu(du) \\
&= \int_0^t \nu(du) \int_0^u k_s(u)p_{s-}(u)dA_s + \int_t^\infty \nu(du) \int_0^t k_s(u)p_{s-}(u)dA_s \\
&= \int_0^t \langle x, p.(u)\rangle_u \ \nu(du) + \int_t^\infty \langle x, p.(u)\rangle_t \ \nu(du) \\
&= \int_0^\infty \langle x, p.(u)\rangle_t \ \nu(du) + \int_0^t \left(\langle x, p.(u)\rangle_u - \langle x, p.(u)\rangle_t\right) \nu(du) \, .
\end{aligned}$$

To verify (8.3.3), it suffices to show that the process $x\mu - \xi$ is an $\mathbb{F}$-local martingale (since ξ is a predictable, finite variation process). By definition, for ν-almost every $u \in \mathbb{R}^+$, the process $(m_t(u) := x_t p_t(u) - \langle x, p.(u)\rangle_t, t \geq 0)$ is an $\mathbb{F}$-local martingale. Then, given that $1 = \int_0^\infty p_t(u)\nu(du)$ for every $t \geq 0$, a.s., we have

$$\begin{aligned}
x_t\mu_t - \xi_t &= x_t \int_0^\infty p_t(u)\nu(du) - x_t \int_0^t (p_t(u) - p_u(u))\, \nu(du) \\
&- \int_0^\infty \langle x, p.(u)\rangle_t \ \nu(du) + \int_0^t \left(\langle x, p.(u)\rangle_t - \langle x, p.(u)\rangle_u\right) \nu(du) \\
&= \int_0^\infty m_t(u)\nu(du) - \int_0^t (m_t(u) - m_u(u))\, \nu(du) + x_t \int_0^t p_u(u)\nu(du) - \int_0^t p_u(u)x_u\nu(du) \, .
\end{aligned}$$

The first two terms are martingales (this follows easily from the martingale property of $m(u)$). As for the last term, using the fact that ν has no atoms, we find

$$\begin{aligned}
& d\left(x_t \int_0^t p_u(u)\nu(du) - \int_0^t p_u(u)x_u\nu(du)\right) \\
&= \left(\int_0^t p_u(u)\nu(du)\right) dx_t + x_t p_t(t)\nu(dt) - p_t(t)x_t\nu(dt) = \left(\int_0^t p_u(u)\nu(du)\right) dx_t
\end{aligned}$$

and we have, indeed, proved that $x\mu - \xi$ is an $\mathbb{F}$-local martingale. □

Proposition 8.3.3 ***Canonical Decomposition in*** $\mathbb{G}$
Any (càdlàg) $(\mathbb{P}, \mathbb{F})$*-local martingale* x *is a* $(\mathbb{P}, \mathbb{G})$ *semi-martingale with canonical decomposition*

$$x_t = X_t + \int_0^{t\wedge\tau} \frac{d\langle x, G\rangle_s}{G_{s-}} + \int_{t\wedge\tau}^t \frac{d\langle x, p.(\tau)\rangle_s}{p_{s-}(\tau)}, \tag{8.3.4}$$

where X *is a* $(\mathbb{P}, \mathbb{G})$*-local martingale.*

PROOF: VOIR CARTHAGE □

Exercise 8.3.4 Give a direct check of Proposition 8.3.3 in a Brownian filtration ◁

We end this section proving that any $(\mathbb{P}^*, \mathbb{G})$-martingale remains a $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$-semimartingale, but it is not necessarily a $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$-martingale. Indeed, we have the following result.

Lemma 8.3.5 *Any* $(\mathbb{P}^*, \mathbb{G})$*-martingale* Y^* *is a* $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$ *semi-martingale which can have a non-null bounded variation part.*

PROOF: The result follows immediately from Proposition 8.2.2 (under $\mathbb{P}^*$), noticing that the $(\mathbb{P}^*, \mathbb{G})$ martingale Y^* can be written as $Y^*_t = \tilde{y}^*_t 1\!\!1_{t<\tau} + \hat{y}^*_t(\tau) 1\!\!1_{\tau\le t}$. Therefore, in the filtration $\mathbb{G}^\tau$, it is the sum of two $\mathbb{G}^\tau$ semi-martingales: the processes $1\!\!1_{t<\tau}$ and $1\!\!1_{\tau\le t}$ are $\mathbb{G}^\tau$ semi-martingales, as well as the processes $\tilde{y}, \hat{y}^*(\tau)$. Indeed, from Proposition 8.2.2, recalling that the $(\mathbb{P}^*, \mathbb{F})$-density of τ is a constant equal to one, we know that, for every $u > 0$, $\big(\hat{y}^*_t(u), t \ge u\big)$ is an $\mathbb{F}$-martingale and that the process $\big(\tilde{y}^*_t G(t) + \int_0^t \hat{y}^*_u(u)\nu(du), t \ge 0\big)$ is an $\mathbb{F}$-martingale, hence $\tilde{y}^*$ is a $\mathbb{G}$-semi-martingale.

It can be noticed that the $(\mathbb{P}^*, \mathbb{G})$-martingale M^*, is such that M^*_t is, for any t, a $\mathcal{G}^\tau_0$-measurable random variable. Therefore, M^* is not a $(\mathbb{P}^*, \mathbb{G}^\tau)$-martingale, since, for $s \le t$, $\mathbb{E}(M^*_t|\mathcal{G}^\tau_s) = M^*_t \neq M^*_s$, but it is a bounded variation $\mathbb{F}^{(\tau)}$-predictable process, hence a $\mathbb{G}^\tau$-semi-martingale with null martingale part. In other terms, $\mathbb{H}$ is not immersed in $\mathbb{G}^\tau$ under $\mathbb{P}^*$. □

As in Lemma 8.3.5, we deduce that any $(\mathbb{P}, \mathbb{G})$-martingale is a $(\mathbb{P}, \mathbb{F}^{(\tau)})$-semi-martingale. Note that this result can also be proved using Lemma 8.3.5 and a change of probability argument: a $(\mathbb{P}, \mathbb{G})$-martingale is a $(\mathbb{P}^*, \mathbb{G})$-semi-martingale (from Girsanov's theorem), thus also a $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$-semi-martingale in view of Lemma 8.3.5. By another use of Girsanov's theorem, it is thus a $(\mathbb{P}, \mathbb{F}^{(\tau)})$-semi-martingale.

8.4 Predictable Representation Theorems

The aim of this section is to obtain Predictable Representation Property (PRP hereafter) in the enlarged filtrations $\mathbb{G}$ and $\mathbb{F}^{(\tau)}$, both under $\mathbb{P}$ and $\mathbb{P}^*$. We start by assuming that there exists a $(\mathbb{P}, \mathbb{F})$-local martingale z (possibly multidimensional), such that the PRP holds in $(\mathbb{P}, \mathbb{F})$. Notice that z is not necessarily continuous.

Beforehand we introduce some notation: $\mathcal{M}_{\text{loc}}(\mathbb{P}, \mathbb{F})$ denotes the set of $(\mathbb{P}, \mathbb{F})$-local martingales, and $\mathcal{M}^2(\mathbb{P}, \mathbb{F})$ denotes the set of $(\mathbb{P}, \mathbb{F})$-martingales x, such that

$$\mathbb{E}\left(x_t^2\right) < \infty, \quad \forall\ t \ge 0. \tag{8.4.1}$$

Also, for a $(\mathbb{P}, \mathbb{F})$-local martingale m, we denote by $\mathcal{L}(m, \mathbb{P}, \mathbb{F})$ the set of $\mathbb{F}$-predictable processes which are integrable with respect to m (in the sense of local martingale), namely (see, e.g., Definition 9.1 and Theorem 9.2. in [59])

$$\mathcal{L}(m, \mathbb{P}, \mathbb{F}) = \left\{ \varphi \in \mathcal{P}(\mathbb{F}) : \left(\int_0^{\cdot} \varphi_s^2 d[m]_s \right)^{1/2} \text{ is } \mathbb{P} - \text{locally integrable} \right\}.$$

Hypothesis 8.4.1 ***PRP for*** $(\mathbb{P}, \mathbb{F})$
There exists a process $z \in \mathcal{M}_{\text{loc}}(\mathbb{P}, \mathbb{F})$ *such that every* $x \in \mathcal{M}_{\text{loc}}(\mathbb{P}, \mathbb{F})$ *can be represented as*

$$x_t = x_0 + \int_0^t \varphi_s dz_s$$

for some $\varphi \in \mathcal{L}(z, \mathbb{P}, \mathbb{F})$.

We start investigating what happens under the measure $\mathbb{P}^*$, in the initially enlarged filtration $\mathbb{F}^{(\tau)}$.

Recall that, assuming the immersion property, Kusuoka [88] has established a PRP for the progressively enlarged filtration, in the case where $\mathbb{F}$ is a Brownian filtration.

Also, under the *equivalence assumption* in $[0, T]$ and assuming a PRP in the reference filtration $\mathbb{F}$, Amendinger (see [5, Th. 2.4]) proved a PRP in $(\mathbb{P}^*, \mathbb{F}^{(\tau)}$ and extended the result to $(\mathbb{P}, \mathbb{F}^{(\tau)})$, in the case where the underlying (local) martingale in the reference filtration is continuous.

Proposition 8.4.2 ***PRP for*** $(\mathbb{P}^*, \mathbb{F}^{(\tau)})$
Under Assumption 8.4.1, every $X^{(\tau)} \in \mathcal{M}_{\text{loc}}(\mathbb{P}^*, \mathbb{F}^{(\tau)})$ *admits a representation*

$$X_t^{(\tau)} = X_0^{(\tau)} + \int_0^t \Phi_s^\tau dz_s \tag{8.4.2}$$

where $\Phi^\tau \in \mathcal{L}(z, \mathbb{P}^*, \mathbb{F}^{(\tau)})$. *In the case where* $X^{(\tau)} \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{F}^{(\tau)})$, *one has* $\mathbb{E}^*\big(\int_0^t (\Phi_s^\tau)^2 d[z]_s\big) < \infty$, *for all* $t \geq 0$ *and the representation is unique.*

Proof: From Theorem 13.4 in [59], it suffices to prove that any bounded martingale admits a predictable representation in terms of z. Let $X^{(\tau)} \in \mathcal{M}_{\text{loc}}(\mathbb{P}^*, \mathbb{F}^{(\tau)})$ be bounded by K. From Proposition 8.2.1, $X_t^{(\tau)} = x_t(\tau)$ where, for ν-almost every $u \in \mathbb{R}^+$, the process $\big(x_t(u), t \geq 0\big)$ is a $(\mathbb{P}^*, \mathbb{F})$-martingale, hence a $(\mathbb{P}, \mathbb{F})$-martingale. Thus Assumption 8.4.1 implies that (for ν-almost every $u \in \mathbb{R}^+$),

$$x_t(u) = x_0(u) + \int_0^t \varphi_s(u) dz_s \ ,$$

where $(\varphi_t(u), t \geq 0)$ is an $\mathbb{F}$-predictable process.

The process $X^{(\tau)}$ being bounded by K, it follows by an application of Lemma 8.1.4(i) that for ν-almost every $u \geq 0$, the process $(x_t(u), t \geq 0)$ is bounded by K. Then, using the Itô isometry,

$$\begin{aligned} \mathbb{E}^*\big(\int_0^t \varphi_s^2(u) d[z]_s\big) &= \mathbb{E}^*\big(\int_0^t \varphi_s(u) dz_s\big)^2 \\ &= \mathbb{E}^*\big((x_t(u) - x_0(u))^2\big) \leq \mathbb{E}^*(x_t^2(u)) \leq K^2 \ . \end{aligned}$$

Also, from [107, Lemma 2], one can consider a version of the process $\int_0^\cdot \varphi_s^2(u) d[z]_s$ which is measurable with respect to u. Using this fact,

$$\mathbb{E}^*\Big[\big(\int_0^t \varphi_s^2(\tau) d[z]_s\big)^{1/2}\Big] = \int_0^\infty \nu(du)\Big(\mathbb{E}^*\big(\int_0^t \varphi_s^2(u) d[z]_s\big)\Big)^{1/2} \leq \int_0^\infty \nu(du) K = K \ .$$

The process $\Phi^{(\tau)}$ defined by $\Phi_t^{(\tau)} = \varphi_t(\tau)$ is $\mathbb{F}^{(\tau)}$-predictable, according to Proposition 5.1.1, it satisfies (8.4.2), with $X_0^{(\tau)} = x_0(\tau)$, and it belongs to $\mathcal{L}(z, \mathbb{P}^*, \mathbb{F}^{(\tau)})$.

If $X^{(\tau)} \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{F}^{(\tau)})$, from Itô's isometry,

$$\mathbb{E}^*\left(\int_0^t (\Phi_s^{(\tau)})^2 d[z]_s\right) = \mathbb{E}^*\left(\int_0^t \Phi_s^{(\tau)} dz_s\right)^2 = \mathbb{E}^*(X_t^{(\tau)} - X_0^{(\tau)})^2 < \infty \ .$$

Also, from this last equation, if $X^{(\tau)} \equiv 0$ then $\Phi^{(\tau)} \equiv 0$, from which the uniqueness of the representation follows. □

Passing to the progressively enlarged filtration $\mathbb{G}$, which consists of two filtrations, $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, intuitively one needs two martingales to establish a PRP. Apart from z, intuition tells us that a candidate for the second martingale might be the compensated martingale of H, that was introduced, respectively under $\mathbb{P}$ (it was denoted by M) and under $\mathbb{P}^*$ (denoted by M^*), in Equation (8.1.5) and in Equation (8.1.6).

Proposition 8.4.3 ***PRP for*** $(\mathbb{P}^*, \mathbb{G})$
Under Assumption 8.4.1, every $X \in \mathcal{M}_{\text{loc}}(\mathbb{P}^*, \mathbb{G})$ *admits a representation*

$$X_t = X_0 + \int_0^t \Phi_s dz_s + \int_0^t \Psi_s dM_s^*$$

for some processes $\Phi \in \mathcal{L}(z, \mathbb{P}^*, \mathbb{G})$ *and* $\Psi \in \mathcal{L}(M^*, \mathbb{P}^*, \mathbb{G})$. *Moreover, if* $X \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{G})$, *one has, for any* te0,

$$\mathbb{E}^*\left(\int_0^t \Phi_s^2 d[z]_s\right) < \infty \quad , \quad \mathbb{E}^*\left(\int_0^t \Psi_s^2 \lambda^*(s)\nu(ds)\right) < \infty \, ,$$

and the representation is unique.

PROOF: It is known that any $(\mathbb{P}^*, \mathbb{H})$ local martingale ξ can be represented as $\xi_t = \xi_0 + \int_0^t \psi_s dM_s^*$ for some process $\psi \in \mathcal{L}(M^*, \mathbb{P}^*, \mathbb{H})$ (see, e.g., the proof in [29]). Notice that ψ has a role only before τ and, for this reason (recall that $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ is the natural filtration of the indicator process H), ψ can be chosen deterministic.

Under $\mathbb{P}^*$, we then have

- the PRP holds in $\mathbb{F}$ with respect to z,
- the PRP holds in $\mathbb{H}$ with respect to M^*,
- the filtration $\mathbb{F}$ and $\mathbb{H}$ are independent.

From classical literature (see Lemma 9.5.4.1(ii) in [28], for instance) the filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ enjoys the PRP under $\mathbb{P}^*$ with respect to the pair (z, M^*).

Now suppose that $X \in \mathcal{M}^2(\mathbb{P}^*, \mathbb{G})$. We find

$$\begin{aligned}\infty \quad > \quad & \mathbb{E}^*(X_t - X_0)^2 = \mathbb{E}^*\left(\int_0^t \Phi_s dz_s + \int_0^t \Psi_s dM_s^*\right)^2 \\ = \quad & \mathbb{E}^*\left(\int_0^t \Phi_s^2 d[z]_s\right) + 2\mathbb{E}^*\left(\int_0^t \Phi_s dz_s \int_0^t \Psi_s dM_s^*\right) + \mathbb{E}^*\left(\int_0^t \Psi_s^2 \lambda^*(s)\nu(ds)\right),\end{aligned}$$

where in the last equality we used the Itô isometry. The cross-product term in the last equality is zero due to the orthogonality of z and M^* (under $\mathbb{P}^*$). From this inequality, the desired integrability conditions hold and the uniqueness of the representation follows (as in the previous proposition). □

Remark 8.4.4 In order to establish a PRP for the initially enlarged filtration $\mathbb{F}^{(\tau)}$ and under $\mathbb{P}^*$, one could have proceeded as in the proof of Proposition 8.4.3, noting that any martingale ξ in the "constant" filtration $\sigma(\tau)$ satisfies $\xi_t = \xi_0 + 0$ and that under $\mathbb{P}^*$ the two filtrations $\mathbb{F}$ and $\sigma(\tau)$ are independent.

Proposition 8.4.5 ***PRP under*** $\mathbb{P}$
Under Assumption 8.4.1, one has:

(i) Every $X^{(\tau)} \in \mathcal{M}_{\text{loc}}(\mathbb{P}, \mathbb{F}^{(\tau)})$ *can be represented as*

$$X_t^{(\tau)} = X_0^{(\tau)} + \int_0^t \Phi_s^{(\tau)} dz_s^{(\tau)}$$

where $z^{(\tau)}$ *is the martingale part in the* $\mathbb{F}^{(\tau)}$*-canonical decomposition of* z *and* $\Phi \in \mathcal{L}(z^{(\tau)}, \mathbb{P}, \mathbb{F}^{(\tau)})$.

(ii) Every $X \in \mathcal{M}_{\text{loc}}(\mathbb{P}, \mathbb{G})$ *can be represented as*

$$X_t = X_0 + \int_0^t \Phi_s dZ_s + \int_0^t \Psi_s dM_s,$$

where Z *is the martingale part in the* $\mathbb{G}$*-canonical decomposition of* z, M *is the* $(\mathbb{P}, \mathbb{G})$*-compensated martingale associated with* H *and* $\Phi \in \mathcal{L}(Z, \mathbb{P}, \mathbb{G})$, $\Psi \in \mathcal{L}(M, \mathbb{P}, \mathbb{G})$.

PROOF: The assertion (i) (resp. (ii)) follows from Proposition 8.4.2 (resp. Proposition 8.4.3) and the stability of PRP under an equivalent change of measure (see for example Theorem 13.12 in [59]). ✓details

For part (ii), it is important to note that, if z is a $(\mathbb{P},\mathbb{F})$-martingale, it is a $(\mathbb{P}^*,\mathbb{G})$-martingale, too. Hence, by a Girsanov type transformation, Z defined as $dZ_t := dz_t - \frac{1}{\ell^*_{t_-}} d\langle z, \ell^*\rangle_t$, $Z_0 = z_0$, is a $(\mathbb{P},\mathbb{G})$-martingale, where $\ell^* := 1/\ell$ is a $(\mathbb{P}^*,\mathbb{G})$-martingale (in fact $d\mathbb{P}_{|\mathcal{G}_t} = \ell^*_t \, d\mathbb{P}^*{}_{|\mathcal{G}_t}$). From the uniqueness of the canonical decomposition of the $(\mathbb{P},\mathbb{G})$-semimartingale z (which is indeed special) and from Proposition 8.3.3, it follows that the $(\mathbb{P},\mathbb{G})$-martingale Z is in particular given by

$$Z_t = z_t - \int_0^{t\wedge\tau} \frac{d\langle z, G\rangle_s}{G_{s_-}} - \int_{t\wedge\tau}^t \frac{d\langle z, p.(\tau)\rangle_s}{p_{s_-}(\tau)} .$$

□

8.5 Change of probability

In this section, we show how the various quantities associated with a random time τ are transformed under a change of probability. We recall that the intensity is the $\mathbb{F}$ adapted process λ such that $H_t - \int_0^{t\wedge\tau} \lambda_s ds$ is a martingale and that the Azéma supermartingale factorizes as $G_t = N_t e^{-\Lambda_t}$.

Theorem 8.5.1 *Let $Z^{\mathbb{G}}_t = z_t \mathbb{1}_{\{\tau>t\}} + z_t(\tau)\mathbb{1}_{\{\tau\le t\}}$ be a positive $\mathbb{G}$-martingale with $Z^{\mathbb{G}}_0 = 1$ and let $Z^{\mathbb{F}}_t = z_t G_t + \int_0^t z_t(u)p_t(u)\nu(du)$ be its $\mathbb{F}$ projection.*
Let $\mathbb{Q}$ be the probability measure defined on $\mathcal{G}_t$ by $d\mathbb{Q} = Z^{\mathbb{G}}_t d\mathbb{P}$. Then,
(i) for $t \ge \theta$ $p^{\mathbb{Q}}_t(\theta) = p_t(\theta)\frac{z_t(\theta)}{Z^{\mathbb{F}}_t}$,
(ii) the $\mathbb{Q}$-Azéma's supermartingale is defined by $G^{\mathbb{Q}}_t = G_t \dfrac{z_t}{Z^{\mathbb{F}}_t}$
(iii) the $(\mathbb{F},\mathbb{Q})$-intensity process is $\lambda^{\mathbb{F},\mathbb{Q}}_t = \lambda^{\mathbb{F}}_t \dfrac{z_t(t)}{z_{t-}}$, dt- a.s.;
(iv) $N^{\mathbb{F},\mathbb{Q}}$ is the $(\mathbb{F},\mathbb{Q})$-local martingale

$$N^{\mathbb{F},\mathbb{Q}}_t = N^{\mathbb{F}}_t \frac{z_t}{Z^{\mathbb{F}}_t} \exp \int_0^t (\lambda^{\mathbb{F},\mathbb{Q}}_s - \lambda^{\mathbb{F}}_s) ds$$

PROOF: From change of probability

$$\mathbb{Q}(\tau > \theta|\mathcal{F}_t) = \frac{1}{\mathbb{E}(Z^{\mathbb{G}}_t|\mathcal{F}_t)} \mathbb{E}_{\mathbb{P}}(Z^{\mathbb{G}}_t \mathbb{1}_{\tau>\theta}|\mathcal{F}_t) = \frac{1}{Z^{\mathbb{F}}_t}\mathbb{E}_{\mathbb{P}}(z_t(\tau)\mathbb{1}_{\tau>\theta}|\mathcal{F}_t) = \frac{1}{Z^{\mathbb{F}}_t}\int_\theta^\infty z_t(u)p_t(u)\nu(du)$$

The form of the survival process follows immediately by differentiation. The form of the intensity is obvious. The form of N is obtained follows from the definition

$$G^{\mathbb{Q}}_t = N^{\mathbb{Q}}_t e^{-\Lambda^{\mathbb{Q}}_t} = G_t \frac{z_t}{Z^{\mathbb{F}}_t} e^{-\Lambda^{\mathbb{P}}_t}$$

□

Girsanov's tranform with Doléans Dade exponential

We restrict our attention to the case where τ is constructed on a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with a given intensity λ as in the Cox process model, where $\mathbb{F}$ is a Brownian filtration generated by W. Any strictly positive martingale can be written as

$$dL_t = L_{t-}(\Psi_t dW_t + \Phi_t dM_t)$$

where Ψ and Ψ are $\mathbb{G}$ predictable processes, of the form

$$\begin{aligned}\Psi_t &= \psi_t 1\!\!1_{t<\tau} + \psi_t(\tau) 1\!\!1_{\tau \le t} \\ \Phi_t &= \phi_t 1\!\!1_{t<\tau} + \phi_t(\tau) 1\!\!1_{\tau \le t}\end{aligned}$$

where ψ and ϕ are $\mathbb{F}$-predictable. It follows that

$$\begin{aligned}L_t &= \exp\left(\int_0^t \psi_s dW_s - \frac{1}{2}\int_0^t \psi_s^2 ds\right) \exp\left(-\int_0^t \lambda_s \gamma_s ds\right) := \widetilde{L}_t,\ t<\tau \\ &= L_{\tau-}(1+\gamma_\tau) \exp\left(\int_\tau^t \psi_s(\tau) dW_s - \frac{1}{2}\int_\tau^t (\psi_s(\tau))^2 ds\right) = L_{\tau-}(1+\gamma_\tau)\Upsilon_t(\tau),\ \tau<t\end{aligned}$$

where $\Upsilon_t(u) = \exp\left(\int_u^t \psi_s(u) dW_s - \frac{1}{2}\int_u^t (\psi_s(u))^2 ds\right)$ Let $d\mathbb{Q} = L_t d\mathbb{P}$. Then, setting

$$\ell_t = \mathbb{E}(L_t|\mathcal{F}_t) = \widetilde{L}_t e^{-\Lambda_t} + \int_0^t \widetilde{L}_u (1+\gamma_u)\Upsilon_t(u)\lambda_u e^{-\Lambda_u} du$$

$$\mathbb{Q}(\tau > \theta|\mathcal{F}_t) = \frac{1}{\ell_t}\left(\widetilde{L}_t e^{-\Lambda_t} + \int_\theta^t \widetilde{L}_u (1+\gamma_u)\Upsilon_t(u)\lambda_u e^{-\Lambda_u} du\right)$$

It remains to differentiate wrt θ

$$\alpha_t(\theta) = \frac{1}{\ell_t}\widetilde{L}_\theta (1+\gamma_\theta)\Upsilon_t(\theta)\lambda_\theta e^{-\Lambda_\theta}$$

Exercise 8.5.2 Prove that the change of probability measure generated by the two processes

$$z_t = (L_t^{\mathbb{F}})^{-1}, \qquad z_t(\theta) = \frac{p_\theta(\theta)}{p_t(\theta)}$$

provides a model where the immersion property holds true, and where the intensity processes does not change $\lhd$

Exercise 8.5.3 Check that

$$\mathbb{E}(\int_0^{t\wedge\tau} \frac{d\langle X, G\rangle_s}{G_{s-}} - \int_{t\wedge\tau}^t \frac{d\langle X, p(\theta)\rangle_s}{p_{s-}(\theta)}\Big|_{\theta=\tau} |\mathcal{F}_t)$$

is an $\mathbb{F}$-martingale.
Check that that

$$\mathbb{E}(\int_0^t \frac{d\langle X, p(\theta)\rangle_s}{p_{s-}(\theta)}\Big|_{\theta=\tau} |\mathcal{G}_t)$$

is a $\mathbb{G}$ martingale. $\lhd$

Exercise 8.5.4 Let λ be a positive $\mathbb{F}$-adapted process and $\Lambda_t = \int_0^t \lambda_s ds$ and Θ be a strictly positive random variable such that there exists a family $\gamma_t(u)$ which satisfies $\mathbb{P}(\Theta > \theta|\mathcal{F}_t) = \int_\theta^\infty \gamma_t(u) du$. Let $\tau = \inf\{t > 0 : \Lambda_t \ge \Theta\}$.Prove that the density of τ is given by

$$p_t(\theta) = \lambda_\theta \gamma_t(\Lambda_\theta) \text{ if } t \ge \theta \qquad \text{and} \qquad p_t(\theta) = \mathbb{E}[\lambda_\theta \gamma_\theta(\Lambda_\theta)|\mathcal{F}_t] \text{ if } t < \theta.$$

Conversely, if we are given a density p, prove that it is possible to construct a threshold Θ such that τ has p as density. $\lhd$

8.5.1 Application: Defaultable Zero-Coupon Bonds

A defaultable zero-coupon with maturity T associated with the default time τ is an asset which pays one monetary unit at time T if (and only if) the default has not occurred before T. We assume that $\mathbb{P}$ is the pricing measure. By definition, the risk-neutral price under $\mathbb{P}$ of the T-maturity defaultable zero-coupon bond with zero recovery equals, for every $t \in [0,T]$,

$$D(t,T) := \mathbb{P}(\tau > T \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \frac{\mathbb{P}(\tau > T \,|\, \mathcal{F}_t)}{G_t} = 1\!\!1_{\{\tau>t\}} \frac{\mathbb{E}_{\mathbb{P}}\big(N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t\big)}{G_t} \tag{8.5.1}$$

where N is the martingale part in the multiplicative decomposition of G (see Proposition). Using (8.5.1), we obtain

$$D(t,T) \;:=\; \mathbb{P}(\tau > T \,|\, \mathcal{G}_t) = 1\!\!1_{\{\tau>t\}} \frac{1}{N_t e^{-\Lambda_t}} \mathbb{E}_{\mathbb{P}}(N_T e^{-\Lambda_T} \,|\, \mathcal{F}_t)$$

However, using a change of probability, one can get rid of the martingale part N, assuming that there exists p such that

$$\mathbb{P}(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty p_t(u)du$$

Let $\mathbb{P}^*$ be defined as

$$d\mathbb{P}^*|_{\mathcal{G}_t} = Z_t^* d\mathbb{P}|_{\mathcal{G}_t}$$

where Z^* is the $(\mathbb{P},\mathbb{G})$-martingale defined as

$$Z_t^* = 1\!\!1_{\{t<\tau\}} + 1\!\!1_{\{t\geq\tau\}} \lambda_\tau e^{-\Lambda_\tau} \frac{N_t}{p_t(\tau)}$$

Note that

$$d\mathbb{P}^*|_{\mathcal{F}_t} = N_t d\mathbb{P}|_{\mathcal{F}_t} = N_t d\mathbb{P}|_{\mathcal{F}_t}$$

and that $\mathbb{P}^*$ and $\mathbb{P}$ coincide on $\mathcal{G}_\tau$.
Indeed,

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(Z_t^*|\mathcal{F}_t) &= G_t + \int_0^t \lambda_u e^{-\Lambda_u} \frac{N_t}{p_t(u)} p_t(u)\eta(du) \\
&= N_t e^{-\Lambda_t} + N_t \int_0^t \lambda_u e^{-\Lambda_u}\eta(du) = N_t e^{-\Lambda_t} + N_t(1-e^{-\Lambda_t})
\end{aligned}$$

Then, for $t > \theta$,

$$\begin{aligned}
\mathbb{P}^*(\theta < \tau|\mathcal{F}_t) &= \frac{1}{N_t}\mathbb{E}_{\mathbb{P}}(Z_t^* 1\!\!1_{\theta<\tau}|\mathcal{F}_t) = \frac{1}{N_t}\mathbb{E}_{\mathbb{P}}(1\!\!1_{t<\tau} + 1\!\!1_{\{t\geq\tau>\theta\}}\lambda_\tau e^{-\Lambda_\tau}\frac{N_t}{p_t(\tau)}|\mathcal{F}_t) \\
&= \frac{1}{N_t}\left(N_t e^{-\Lambda_t} + \int_\theta^t \lambda_u e^{-\Lambda_u}\frac{N_t}{p_t(u)}p_t(u)du\right) \\
&= \frac{1}{N_t}\left(N_t e^{-\Lambda_t} + N_t(e^{-\Lambda_\theta} - e^{-\Lambda_t})\right) = e^{-\Lambda_\theta}
\end{aligned}$$

which proves that immersion holds true under $\mathbb{P}^*$, and the intensity of τ is the same under $\mathbb{P}$ and $\mathbb{P}^*$. It follows that

$$\mathbb{E}_{\mathbb{P}}(X 1\!\!1_{\{T<\tau\}}|\mathcal{G}_t) = \mathbb{E}^*(X 1\!\!1_{\{T<\tau\}}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}}\frac{1}{e^{-\Lambda_t}}\mathbb{E}^*(e^{-\Lambda_T}X|\mathcal{F}_t)$$

Note that, if the intensity is the same under $\mathbb{P}$ and $\mathbb{P}^*$, its dynamics under $\mathbb{P}^*$ will involve a change of driving process, since $\mathbb{P}$ and $\mathbb{P}^*$ do not coincide on $\mathcal{F}_\infty$.

Let us now study the pricing of a recovery. Let Z be an $\mathbb{F}$-predictable bounded process.

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(Z_\tau 1\!\!1_{\{t<\tau\leq T\}}|\mathcal{G}_t) &= 1\!\!1_{\{t<\tau\}}\frac{1}{G_t}\mathbb{E}_{\mathbb{P}}(-\int_t^T Z_u dG_u|\mathcal{F}_t)\\
&= 1\!\!1_{\{t<\tau\}}\frac{1}{G_t}\mathbb{E}_{\mathbb{P}}(\int_t^T Z_u N_u\lambda_u e^{-\Lambda_u}du|\mathcal{F}_t)\\
&= \mathbb{E}^*(Z_\tau 1\!\!1_{\{t<\tau\leq T\}}|\mathcal{G}_t)\\
&= 1\!\!1_{\{t<\tau\}}\frac{1}{e^{-\Lambda_t}}\mathbb{E}^*(\int_t^T Z_u\lambda_u e^{-\Lambda_u}du|\mathcal{F}_t)
\end{aligned}$$

The problem is more difficult for pricing a recovery paid at maturity, i.e. for $X\in\mathcal{F}_T$

$$\begin{aligned}
\mathbb{E}_{\mathbb{P}}(X1\!\!1_{\tau<T}|\mathcal{G}_t) &= \mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t)-\mathbb{E}_{\mathbb{P}}(X1\!\!1_{\tau>T}|\mathcal{G}_t)=\mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t)-1\!\!1_{\{\tau>t\}}\frac{1}{N_te^{-\Lambda_t}}\mathbb{E}_{\mathbb{P}}\big(XN_Te^{-\Lambda_T}\,|\,\mathcal{F}_t\big)\\
&= \mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t)-1\!\!1_{\{\tau>t\}}\frac{1}{e^{-\Lambda_t}}\mathbb{E}^*\big(Xe^{-\Lambda_T}\,|\,\mathcal{F}_t\big)
\end{aligned}$$

Since immersion holds true under $\mathbb{P}^*$

$$\begin{aligned}
\mathbb{E}^*(X1\!\!1_{\tau<T}|\mathcal{G}_t) &= \mathbb{E}^*(X|\mathcal{G}_t)-1\!\!1_{\{\tau>t\}}\frac{1}{e^{-\Lambda_t}}\mathbb{E}^*\big(XN_Te^{-\Lambda_T}\,|\,\mathcal{F}_t\big)\\
&= \mathbb{E}^*(X|\mathcal{F}_t)-1\!\!1_{\{\tau>t\}}\frac{1}{e^{-\Lambda_t}}\mathbb{E}^*\big(XN_Te^{-\Lambda_T}\,|\,\mathcal{F}_t\big)
\end{aligned}$$

If both quantities $\mathbb{E}_{\mathbb{P}}(X1\!\!1_{\tau<T}|\mathcal{G}_t)$ and $\mathbb{E}^*(X1\!\!1_{\tau<T}|\mathcal{G}_t)$ are the same, this would imply that $\mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t)=\mathbb{E}^*(X|\mathcal{F}_t)$ which is impossible: this would lead to $\mathbb{E}_{\mathbb{P}}(X|\mathcal{G}_t)=\mathbb{E}_{\mathbb{P}}(X|\mathcal{F}_t)$, i.e. immersion holds under P. Hence, non-immersion property is important while evaluating recovery paid at maturity ($\mathbb{P}^*$ and $\mathbb{P}$ do not coincide on $\mathcal{F}_\infty$).

8.6 Forward intensity

By using the density approach, we adopt an additive point of view to represent the conditional probability of τ: the conditional survival function $G_t(\theta)=\mathbb{P}(\tau>\theta\,|\,\mathcal{F}_t)$ is written in the form $G_t(\theta)=\int_\theta^\infty p_t(u)\nu(du)$. In the default framework, the "intensity" point of view is often preferred, and one uses the multiplicative representation $G_t(\theta)=\exp(-\int_0^\theta\lambda_t(u)\nu(du))$. In the particular case where ν denotes the Lebesgue measure (in that case, the law of τ is $p_0(u)$, and we shall), the family of $\mathcal{F}_t$-measurable random variables $\lambda_t(\theta)=-\partial_\theta\ln G_t(\theta)$ is called the "forward intensity". We shall discuss and compare these two points of view further on.

We now consider $(G_t(\theta),t\geq 0)$ as in the classical HJM models where its dynamics is given in multiplicative form. By using the forward intensity $\lambda_t(\theta)$ of τ, the density can then be calculated as $p_t(\theta)=\lambda_t(\theta)G_t(\theta)$. It follows that the forward intensity is non-negative. As noted before, $\lambda(\theta)$ plays the same role as the spot forward rate in the interest rate models.

Proposition 8.6.1 *Let $dG_t(\theta)=Z_t(\theta)dW_t$ be the martingale representation of $(G_t(\theta),t\geq 0)$ and assume that the processes $(Z_t(\theta);t\geq 0)$ are differentiable in the following sense: there exists a family of processes $(z_t(\theta),t\geq 0)$ such that $Z_t(\theta)=\int_0^\theta z_t(u)\nu(du)$, $Z_t(0)=0$. Then, under regularity conditions,*

1) the density processes have the following dynamics $dp_t(\theta)=-z_t(\theta)dW_t$ where $z(\theta)$ is subjected to the constraint $\int_0^\infty z_t(\theta)\nu(d\theta)=0$ for any $t\geq 0$.

2) The survival process G evolves as $dG_t=-\alpha_t(t)\nu(dt)+Z_t(t)dW_t$.

3) With more regularity assumptions, if $(\partial_\theta p_t(\theta))_{\theta=t}$ is simply denoted by $\partial_\theta p_t(t)$, then the process $p_t(t)$ follows :

$$dp_t(t)=\partial_\theta p_t(t)\nu(dt)-z_t(t)dW_t.$$

PROOF: 1) Observe that $Z(0) = 0$ since $G(0) = 1$, hence the existence of z is related with some smoothness conditions. Then using the stochastic Fubini theorem , one has

$$G_t(\theta) = G_0(\theta) + \int_0^t Z_u(\theta)dW_u = G_0(\theta) + \int_0^\theta \nu(dv)\int_0^t z_u(v)dW_u.$$

So 1) follows. Using the fact that for any $t \geq 0$,

$$1 = \int_0^\infty p_t(u)\nu(du) = \int_0^\infty \nu(du)\big(P_0(u) - \int_0^t z_s(u)dW_s\big) = 1 - \int_0^t dW_s \int_0^\infty z_s(u)\nu(du),$$

one gets $\int_0^\infty z_t(u)\nu(du) = 0$.
2) By using Proposition **??** and integration by parts,

$$M_t^{\mathbb{F}} = -\int_0^t (p_t(u) - p_u(u))\nu(du) = \int_0^t \nu(du)\int_u^t z_s(u)dW_s = \int_0^t dW_s\Big(\int_0^s z_s(u)\nu(du)\Big),$$

which implies 2).
3) We follow the same way as for the decomposition of G, by studying the process

$$p_t(t) - \int_0^t (\partial_\theta p_s)(s)\nu(ds) = p_t(0) + \int_0^t (\partial_\theta p_t)(s)\nu(ds) - \int_0^t (\partial_\theta p_s)(s)\nu(ds)$$

where the notation $\partial_\theta p_t(t)$ is defined in 3). Using the martingale representation of $p_t(\theta)$ and integration by parts (assuming that smoothness hypothesis allows these operations), the integral in the RHS is a stochastic integral,

$$\begin{aligned}\int_0^t \Big((\partial_\theta p_t)(s) - (\partial_\theta p_s)(s)\Big)\nu(ds) &= -\int_0^t \nu(ds)\partial_\theta(\int_s^t z_u(\theta)dW_u)\\ = -\int_0^t \nu(ds)\int_s^t \partial_\theta z_u(s)dW_u &= -\int_0^t dW_u \int_0^u \nu(ds)\partial_\theta z_u(s) = -\int_0^t dW_u(z_u(u) - z_u(0))\end{aligned}$$

The stochastic integral $\int_0^t z_u(0)dW_u$ is the stochastic part of the martingale $p_t(0)$, and so the property 3) holds true. □

Classically, HJM framework is studied for time smaller than maturity, i.e. $t \leq T$. Here we consider all positive pairs (t, θ).

Proposition 8.6.2 *We keep the notation and the assumptions in Proposition 8.6.1. For any $t, \theta \geq 0$, let $\Psi_t(\theta) = \frac{Z_t(\theta)}{G_t(\theta)}$. We assume that there exists a family of processes ψ such that $\Psi_t(\theta) = \int_0^\theta \psi_t(u)\nu(du)$. Then*

1) $G_t(\theta) = G_0(\theta)\exp\Big(\int_0^t \Psi_s(\theta)dW_s - \frac{1}{2}\int_0^t |\Psi_s(\theta)|^2 ds\Big)$;

2) the forward intensity $\lambda(\theta)$ has the following dynamics:

$$\lambda_t(\theta) = \lambda_0(\theta) - \int_0^t \psi_s(\theta)dW_s + \int_0^t \psi_s(\theta)\Psi_s(\theta)ds; \tag{8.6.1}$$

3) $S_t = \exp\Big(-\int_0^t \lambda_s^{\mathbb{F}}\nu(ds) + \int_0^t \Psi_s(s)dW_s - \frac{1}{2}\int_0^t |\Psi_s(s)|^2 ds\Big)$;

PROOF: By choice of notation, 1) holds since the process $G_t(\theta)$ is the solution of the equation

$$\frac{dG_t(\theta)}{G_t(\theta)} = \Psi_t(\theta)dW_t, \qquad \forall\, t, \theta \geq 0. \tag{8.6.2}$$

2) is the consequence of 1) and the definition of $\lambda(\theta)$.
3) This representation is the multiplicative version of the additive decomposition of G in Proposition 8.6.1. We recall that $\lambda_t^{\mathbb{F}} = p_t(t)G_t^{-1}$. There are no technical difficulties because G is continuous. □

8.7 Multidefault

Lemma 8.7.1 *Assume that* $\mathbb{P}(\tau_i > t_i, i = 1, \dots, n|\mathcal{F}_t) = \int_{t_1}^{\infty} \dots \int_{t_n}^{\infty} g_t(u_1, \dots, u_n) du_1 \dots du_n$. *Prove that* $\mathbb{F}$ *is immersed in* $\mathbb{G}$ *if and only if* $g_t(t_1, \dots, t_n) = g_u(t_1, \dots, t_n)$ *for* $u > t > \max(t_i)$

8.8 Concluding Remarks

- In the multi-dimensional case, that is when $\tau = (\tau_1, \dots, \tau_d)$ is a vector of finite random times, the same machinery can be applied. More precisely, under the assumption

$$\mathbb{P}(\tau_1 \in d\theta_1, \dots, \tau_d \in \ d\theta_d | \mathcal{F}_t) \sim \mathbb{P}(\tau_1 \in d\theta_1, \dots, \tau_d \in d\theta_d)$$

one defines the probability $\mathbb{P}^*$ equivalent to $\mathbb{P}$ on $\mathcal{F}_t^{(\tau)} = \mathcal{F}_t \vee \sigma(\tau_1) \vee \dots \vee \sigma(\tau_d)$ by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}_{|\mathcal{F}_t^{(\tau)}} = \frac{1}{p_t(\tau_1, \dots, \tau_d)},$$

where $p_t(\tau_1, \dots, \tau_d)$ is the (multidimensional) analog to $p_t(\tau)$, and the results for the initially enlarged filtration are obtained in the same way as for the one-dimensional case.

As for the progressively enlarged filtration $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\tau_1 \wedge t) \vee \dots \vee \sigma(\tau_d \wedge t)$, one has to note that, in this case, a measurable process is decomposed into 2^d terms, corresponding to the measurability of the process on the various sets $\{\tau_i \le t < \tau_j, i \in I, j \in I^c\}$ for all the subsets I of $\{1, ..., d\}$.

- In this study, *honest* times are automatically excluded, as we explain now. Under the probability $\mathbb{P}^*$, the Azéma supermartingale associated with τ being a continuous decreasing function, it has a trivial Doob-Meyer decomposition $G^* = 1 - A^*$ with $A_t^* = \int_0^t \nu(du)$. So, $A_\infty^* = 1$ and, in particular, τ can not be an honest time: recall that in our setting, τ avoids the $\mathbb{F}$-stopping times and therefore, from a result due to Azéma [11], if τ is an honest time, the random variable A_∞^* should have exponential law with parameter 1, which is not the case (note that the notion of honest time does not depend on the probability measure).

- Under immersion property and under the ($\mathcal{E}$)-Hypothesis, $p_t(u) = p_u(u), t \ge u$. In particular, as expected, the canonical decomposition's formulae presented in Section 8.3 are trivial, i.e., the "drift" terms vanish.

- Predictable representation theorems can be obtained in the more general case, where any $(\mathbb{P}, \mathbb{F})$-martingale x admits a representation as

$$x_t = x_0 + \int_0^t \int_E \varphi(s, \theta) \tilde{\mu}(ds, d\theta),$$

for a compensated martingale associated with a point process.

Chapter 9

Conditional Laws of Random Times

9.1 Density Models

In this section, we are interested in densities on $I\!R_+$ in order to give models for the conditional law of a random time τ: more precisely, our goal is to give exemples of processes $g(u)$ so that one can construct a random time τ satisfying $\mathbb{P}(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty g_t(u)du$. We recall the classical construction of default times as first hitting time of a barrier, independent of the reference filtration, and we extend this construction to the case where the barrier is no more independent of the reference filtration. It is then natural to characterize the dependence of this barrier and the filtration by means of its conditional law.

In the literature on credit risk modeling, the attention is mostly focused on the intensity process, i.e., to the process Λ such that $1_{\tau\le t} - \Lambda_{t\wedge\tau}$ is a $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$-martingale, where $\mathcal{H}_t = \sigma(t \wedge \tau)$. We recall that the intensity process Λ is the only increasing predictable process such that the survival process $G_t := P(\tau > t|\mathcal{F}_t)$ admits the decomposition $G_t = N_t e^{-\Lambda_t}$ where N is a local martingale. We recall that the intensity process can be recovered form the density process as $d\Lambda_s = \frac{g_s(s)}{G_s(s)}ds$ (see Proposition 8.1.6). We end the section giving an explicit example of two different martingale survival processes having the same survival processes (hence the intensities are equal).

9.1.1 Structural and reduced-form models

In the literature, models for default times are often based on a threshold: the default occurs when some driving process X reaches a given barrier. Based on this observation, we consider the random time on $I\!R_+$ in a general threshold model. Let X be a stochastic process and Θ be a barrier which we shall precise later. Define the random time as the first passage time

$$\tau := \inf\{t \,:\, X_t \ge \Theta\}\,.$$

In classical structural models, the process X is an $\mathbb{F}$-adapted process associated with the value of a firm and the barrier Θ is a constant. So, τ is an $\mathbb{F}$-stopping time. In this case, the conditional distribution of τ does not have a density process, since $P(\tau > \theta|\mathcal{F}_t) = 1_{\theta<\tau}$ for $\theta \le t$.

To obtain a density process, the model has to be changed, for example one can stipulate that the driving process X is not observable and that the observation is a filtration $\mathbb{F}$, smaller than the filtration $\mathbb{F}^X$, or a filtration including some noise. The goal is again to compute the conditional law of the default $P(\tau > \theta|\mathcal{F}_t)$, using for example filtering theory.

Another method is to consider a right-continuous $\mathbb{F}$-adapted increasing process Γ and to randomize the barrier. The easiest way is to take the barrier Θ as an $\mathcal{A}$-measurable random variable independent of $\mathbb{F}$, and to consider

$$\tau := \inf\{t \,: \Gamma_t \ge \Theta\}\,. \tag{9.1.1}$$

If Γ is continuous, τ is the inverse of Γ taken at Θ, and $\Gamma_\tau = \Theta$. The $\mathbb{F}$-conditional law of τ is

$$P(\tau > \theta|\mathcal{F}_t) = G^\Theta(\Gamma_\theta),\ \theta \le t$$

where G^Θ is the survival probability of Θ given by $G^\Theta(t) = P(\Theta > t)$. We note that in this particular case, $P(\tau > \theta|\mathcal{F}_t) = P(\tau > \theta|\mathcal{F}_\infty)$ for any $\theta \le t$, which means that the H-hypothesis is satisfied and that the martingale survival processes remain constant after θ (i.e., $G_t(\theta) = G_\theta(\theta)$ for $t \ge \theta$). This result is stable by increasing transformation of the barrier, so that we can assume without loss of generality that the barrier is the standard exponential random variable $-\log(G^\Theta(\Theta))$.

If the increasing process Γ is assumed to be absolutely continuous w.r.t. the Lebesgue measure with Radon-Nikodým density γ and if G^Θ is differentiable, then the random time τ admits a density process given by

$$\begin{aligned} g_t(\theta) &= -(G^\Theta)'(\Gamma_\theta)\gamma_\theta = g_\theta(\theta),\ \theta \le t \\ &= E(g_\theta(\theta)|\mathcal{F}_t),\ \theta > t. \end{aligned} \tag{9.1.2}$$

Example (Cox process model) In the widely used Cox process model, the independent barrier Θ follows the exponential law and $\Gamma_t = \int_0^t \gamma_s ds$ represents the default compensator process. As a direct consequence of (9.1.2),

$$g_t(\theta) = \gamma_\theta e^{-\Gamma_\theta},\ \theta \le t.$$

✓ TO BE COMPLETED

9.1.2 Generalized threshold models

In this subsection, we relax the assumption that the threshold Θ is independent of $\mathcal{F}_\infty$. We assume that the barrier Θ is a strictly positive random variable whose conditional distribution w.r.t. $\mathbb{F}$ admits a density process, i.e., there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(I\!\!R_+)$-measurable functions $p_t(u)$ such that

$$G_t^\Theta(\theta) := P(\Theta > \theta|\mathcal{F}_t) = \int_\theta^\infty p_t(u)du\,. \tag{9.1.3}$$

We assume in addition that the process Γ is absolutely continuous w.r.t. the Lebesgue measure, i.e., $\Gamma_t = \int_0^t \gamma_s ds$. We still consider τ defined as in (9.1.1) by $\tau = \Gamma^{-1}(\Theta)$ and we say that a random time constructed in such a setting is given by *a generalized threshold*.

Proposition 9.1.1 *Let τ be given by a generalized threshold. Then τ admits the density process $g(\theta)$ where*

$$g_t(\theta) = \gamma_\theta p_t(\Gamma_\theta),\ \theta \le t. \tag{9.1.4}$$

PROOF: By definition and by the fact that Γ is strictly increasing and absolutely continuous, we have for $t \ge \theta$,

$$G_t(\theta) := P(\tau > \theta|\mathcal{F}_t) = P(\Theta > \Gamma_\theta|\mathcal{F}_t) = G_t^\Theta(\Gamma_\theta) = \int_{\Gamma_\theta}^\infty p_t(u)du = \int_\theta^\infty p_t(\Gamma_u)\gamma_u du,$$

which implies $g_t(\theta) = \gamma_\theta p_t(\Gamma_\theta)$ for $t \ge \theta$.

Obviously, in the particular case where the threshold Θ is independent of $\mathcal{F}_\infty$, we recover the classical results (9.1.2) recalled above.

Conversely, if we are given a density process g, then it is possible to construct a random time τ by a generalized threshold, that is, to find Θ such that the associated τ has g as density, as we show now. It suffices to define $\tau = \inf\{t\ :\ t \ge \Theta\}$ where Θ is a random variable with conditional density $p_t = g_t$. Of course, for any increasing process Γ, $\tau = \inf\{t\ :\ \Gamma_t \ge \Delta\}$ where $\Delta := \Gamma_\Theta$ is a different way to obtain a solution!

9.2 Examples of Densities

The goal of this section is to give examples of the conditional law of a random variable (or a random vector), given a reference filtration, and methods to construct dynamics of conditional laws, in order to model price processes and/or default risk.

We first present two specific examples of conditional law of an $\mathcal{F}^B_\infty$-measurable random variable, when $\mathbb{F}^B$ is the natural filtration of a Brownian motion B. Then we give two large classes of examples, based on Markov processes and diffusion processes.

The first example, despite its simplicity, will allow us to construct a dynamic copula, in a Gaussian framework; more precisely, we construct, for any t, the (conditional) copula of a family of random times $P(\tau_i > t_i, i = 1, \dots, n|\mathcal{F}_t)$ and we can chose the parameters so that $P(\tau_i > t_i, i = 1, \dots, n)$ equals a given (static) Gaussian copula. To the best of our knowledge, there are very few explicit constructions of such a model.
In Fermanian and Vigneron [48], the authors apply a copula methodology, using a factor Y. However, the processes they use to fit the conditional probabilities $P(\tau_i > t_i, i = 1, \dots, n|\mathcal{F}_t \vee \sigma(Y))$ are not martingales. They show that, using some adequate parametrization, they can produce a model so that $P(\tau_i > t_i, i = 1, \dots, n|\mathcal{F}_t)$ are martingales. Our model will satisfy both martingale conditions.
In [27], Carmona is interested in the dynamics of prices of assets corresponding to a payoff which is a Bernoulli random variable (taking values 0 or 1), in other words, he is looking for examples of dynamics of martingales valued in $[0, 1]$, with a given terminal condition. Surprisingly, the example he provides corresponds to the one we gave in Section 5.4.4, up to a particular choice of the parameters to satisfy the terminal constraint.
In a second example, we construct another dynamic copula, again in an explicit way, with a more complicated dependence.

We then show that a class of examples can be obtained from a Markov model, where the decreasing property is introduced via a change of variable. In the second class of examples, the decreasing property is modeled via the dependence of a diffusion through its initial condition. To close the loop, we show that we can recover the Gaussian model of the first example within this framework.

9.2.1 A dynamic Gaussian copula model

In this subsection, φ is the standard Gaussian probability density, and Φ the Gaussian cumulative function. We recall the results obtained in Section 5.4.4.

We consider the random variable $X := \int_0^\infty f(s)dB_s$ where f is a deterministic, square-integrable function. For any real number θ and any positive t, one has

$$P(X > \theta|\mathcal{F}^B_t) = P\Big(m_t > \theta - \int_t^\infty f(s)dB_s|\mathcal{F}^B_t\Big)$$

where $m_t = \int_0^t f(s)dB_s$ is $\mathcal{F}^B_t$-measurable. The random variable $\int_t^\infty f(s)dB_s$ follows a centered Gaussian law with variance $\sigma^2(t) = \int_t^\infty f^2(s)ds$ and is independent of $\mathcal{F}^B_t$. Assuming that $\sigma(t)$ does not vanish, one has

$$P(X > \theta|\mathcal{F}^B_t) = \Phi\Big(\frac{m_t - \theta}{\sigma(t)}\Big) . \tag{9.2.1}$$

In other words, the conditional law of X given $\mathcal{F}^B_t$ is a Gaussian law with mean m_t and variance $\sigma^2(t)$. We summarize the result in the following proposition, and we give the dynamics of the martingale survival process, obtained with a standard use of Itô's rule.

Proposition 9.2.1 *Let B be a Brownian motion, f an L^2 deterministic function, $m_t = \int_0^t f(s)dB_s$*

and $\sigma^2(t) = \int_t^\infty f^2(s)ds$. *The family*

$$G_t(\theta) = \Phi\Big(\frac{m_t - \theta}{\sigma(t)}\Big)$$

is a family of $\mathbb{F}^B$*-martingales, valued in* $[0,1]$*, which is decreasing w.r.t.* θ. *Moreover*

$$dG_t(\theta) = \varphi\Big(\frac{m_t - \theta}{\sigma(t)}\Big)\frac{f(t)}{\sigma(t)}dB_t\,.$$

The dynamics of the martingale survival process can be written

$$dG_t(\theta) = \varphi\left(\Phi^{-1}(G_t(\theta))\right)\frac{f(t)}{\sigma(t)}\,dB_t\,. \tag{9.2.2}$$

We obtain the associated density family by differentiating $G_t(\theta)$ w.r.t. θ,

$$g_t(\theta) = \frac{1}{\sqrt{2\pi}\,\sigma(t)}\,\exp\Big(-\frac{(m_t-\theta)^2}{2\sigma^2(t)}\Big)$$

and its dynamics

$$dg_t(\theta) \quad = \quad -g_t(\theta)\frac{m_t - \theta}{\sigma^2(t)}f(t)dB_t. \tag{9.2.3}$$

In order to provide conditional survival probabilities for positive random variables, we consider $\widetilde{X} = \psi(X)$ where ψ is a differentiable, positive and strictly increasing function and let $h = \psi^{-1}$. The conditional law of $\widetilde{X}$ is

$$\tilde{G}_t(\theta) = \Phi\Big(\frac{m_t - h(\theta)}{\sigma(t)}\Big).$$

We obtain

$$\widetilde{g}_t(\theta) = \frac{1}{\sqrt{2\pi}\sigma(t)}h'(\theta)\,\exp\Big(-\frac{(m_t - h(\theta))^2}{2\sigma^2(t)}\Big)$$

and

$$\begin{aligned} d\widetilde{G}_t(\theta) &= \varphi\Big(\frac{m_t - h(\theta)}{\sigma(t)}\Big)\frac{f(t)}{\sigma(t)}dB_t\,, \\ d\widetilde{g}_t(\theta) &= -\widetilde{g}_t(\theta)\frac{m_t - h(\theta)}{\sigma(t)}\frac{f(t)}{\sigma(t)}dB_t. \end{aligned}$$

✓ FORWRD INTENSITIES, SHOW THAT IF $dG_t(\theta) = G_t(\theta)\sigma_t(\theta)dB_t$ THEN $dn_t = n_t\sigma_t(t)dB_t$, HNAGE THE NOTATION (DELETE tilde)

Introducing an n-dimensional standard Brownian motion $B = (B^i, i = 1,\dots,n)$ and a factor Y, independent of $\mathbb{F}^B$, gives a dynamic copula approach, as we present now. For h_i an increasing function, mapping $I\!R^+$ into $I\!R$, and setting $\tau_i = (h_i)^{-1}(\sqrt{1-\rho_i^2}\int_0^\infty f_i(s)dB_s^i + \rho_i Y)$, for $\rho_i \in (-1,1)$, an immediate extension of the Gaussian model leads to

$$P(\tau_i > t_i, \forall i = 1,\dots,n|\mathcal{F}_t^B \vee \sigma(Y)) = \prod_{i=1}^n \Phi\left(\frac{1}{\sigma_i(t)}\left(m_t^i - \frac{h_i(t_i) - \rho_i Y}{\sqrt{1-\rho_i^2}}\right)\right)$$

where $m_t^i = \int_0^t f_i(s)dB_s^i$ and $\sigma_i^2(t) = \int_t^\infty f_i^2(s)ds$. It follows that

$$P(\tau_i > t_i, \forall i = 1,\dots,n|\mathcal{F}_t^B) = \int_{-\infty}^\infty \prod_{i=1}^n \Phi\left(\frac{1}{\sigma_i(t)}\left(m_t^i - \frac{h_i(t_i) - \rho_i y}{\sqrt{1-\rho_i^2}}\right)\right) f_Y(y)dy\,.$$

Note that, in that setting, the random times $(\tau_i, i = 1, \dots, n)$ are conditionally independent given $\mathbb{F}^B \vee \sigma(Y)$, a useful property which is not satisfied in Fermanian and Vigneron model. For $t = 0$, choosing f_i so that $\sigma_i(0) = 1$, and Y with a standard Gaussian law, we obtain

$$P(\tau_i > t_i, \forall i = 1, \dots, n) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Phi\left(-\frac{h_i(t_i) - \rho_i y}{\sqrt{1-\rho_i^2}}\right) \varphi(y) dy$$

which corresponds, by construction, to the standard Gaussian copula ($h_i(\tau_i) = \sqrt{1-\rho_i^2} X_i + \rho_i Y$, where X_i, Y are independent standard Gaussian variables).

Relaxing the independence condition on the components of the process B leads to more sophisticated examples.

9.2.2 A Gamma model

In the case of Gamma model of Section 5.4.3 where

$$G_t(\theta) = P(A_\infty^{(-\mu)} > \theta | \mathcal{F}_t^B) = \Upsilon\Big(\frac{\theta - A_t^{(-\mu)}}{e^{2B_t^{(-\mu)}}}\Big) 1_{\theta > A_t^{(-\mu)}} + 1_{\theta \leq A_t^{(-\mu)}}.$$

This gives a family of martingale survival processes G, similar to (9.2.4), with gamma structure. It follows that, on $\{\theta > A_t^{(-\mu)}\}$

$$dG_t(\theta) = \frac{1}{2^{\mu-1}\Gamma(\mu)} e^{-\frac{1}{2} Z_t(\theta)} (Z_t(\theta))^\mu dB_t$$

where $Z_t(\theta) = \frac{e^{2B_t^{(-\mu)}}}{\theta - A_t^{(-\mu)}}$ (to have light notation, we do not specify that this process Z depends on μ). One can check that $G_t(\cdot)$ is differentiable w.r.t. θ, so that $G_t(\theta) = \int_\theta^\infty g_t(u) du$, where

$$g_t(\theta) = 1_{\theta > A_t^{(-\mu)}} \frac{1}{2^\mu \Gamma(\mu)} (Z_t(\theta))^{\mu+1} e^{-\frac{1}{2} Z_t(\theta) - 2B_t^{(-\mu)}}.$$

Again, introducing an n-dimensional Brownian motion, a factor Y and the r.vs $\alpha_i A_\infty^{(-\mu,i)} + \rho_i Y$, where α_i and ρ_i are constants, will give an example of a dynamic copula.

9.2.3 Markov processes

Let X be a real-valued Markov process with transition probability $p_T(t, x, y) dy = P(X_T \in dy | X_t = x)$, and Ψ a family of functions $I\!R \times I\!R \to [0,1]$, decreasing w.r.t. the second variable, such that

$$\Psi(x, -\infty) = 1, \Psi(x, \infty) = 0\,.$$

Then, for any T,

$$G_t(\theta) := E(\Psi(X_T, \theta) | \mathcal{F}_t^X) = \int_{-\infty}^{\infty} p_T(t, X_t, y) \Psi(y, \theta) dy$$

is a family of martingale survival processes on $I\!R$. While modeling $(T; x)$-bond prices, Filipovic et al. [49] have used this approach in an affine process framework. See also Keller-Ressel et al. [82].

Example 9.2.2 Let X be a Brownian motion, and $\Psi(x, \theta) = e^{-\theta x^2} \mathbf{1}_{\theta \geq 0} + \mathbf{1}_{\theta \leq 0}$. We obtain a martingale survival process on $\mathbb{R}_+$, defined for $\theta \geq 0$ and $t < T$ as,

$$G_t(\theta) = E\big[\exp(-\theta X_T^2) | \mathcal{F}_t^X\big] = \frac{1}{\sqrt{1 + 2(T-t)\theta}} \exp\big(-\frac{\theta X_t^2}{1 + 2(T-t)\theta}\big)$$

The construction given above provides a martingale survival process $G(\theta)$ on the time interval $[0,T]$. Using a (deterministic) change of time, one can easily deduce a martingale survival process on the whole interval $[0,\infty[$: setting

$$\hat{G}_t(\theta) = G_{h(t)}(\theta)$$

for a differentiable increasing function h from $[0,\infty]$ to $[0,T]$, and assuming that $dG_t(\theta) = G_t(\theta)K_t(\theta)dB_t, t < T$, one obtains

$$d\hat{G}_t(\theta) = \hat{G}_t(\theta)K_{h(t)}(\theta)\sqrt{h'(t)}dW_t$$

where W is a Brownian motion.
One can also randomize the terminal date and consider T as an exponential random variable independent of $\mathbb{F}$. Noting that the previous $G_t(\theta)$'s depend on T, one can write them as $G_pt(\theta,T)$ and consider

$$\widetilde{G}_t(\theta) = \int_0^\infty G_t(\theta,z)e^{-z}dz$$

which is a martingale survival process. The same construction can be done with a random time T with any given density, independent of $\mathbb{F}$.

9.2.4 Diffusion-based model with initial value

Lemma 9.2.3 *Let Ψ be a cumulative distribution function of class C^2, and Y the solution of*

$$dY_t = a(Y_t)dt + \nu(Y_t)dB_t,\ Y_0 = y_0$$

where a and ν are deterministic functions smooth enough to ensure that the solution of the above SDE is unique. Then, the process $(\Psi(Y_t), t \geq 0)$ is a martingale, valued in $[0,1]$, if and only if

$$a(y)\Psi'(y) + \frac{1}{2}\nu^2(y)\Psi''(y) = 0\,. \tag{9.2.4}$$

Proof: The result follows by applying Itô's formula and noting that $\Psi(Y_t)$ being a (bounded) local martingale is a martingale.

We denote by $Y_t(y)$ the solution of the above SDE with initial condition $Y_0 = y$. Note that, from the uniqueness of the solution, $y \to Y_t(y)$ is increasing (i.e., $y_1 > y_2$ implies $Y_t(y_1) \geq Y_t(y_2)$). It follows that

$$G_t(\theta) := 1 - \Psi(Y_t(\theta))$$

is a family of martingale survival processes.

Example 9.2.4 Let us reduce our attention to the case where Ψ is the cumulative distribution function of a standard Gaussian variable. Using the fact that $\Phi''(y) = -y\Phi'(y)$, Equation (9.2.4) reduces to

$$a(t,y) - \frac{1}{2}y\nu^2(t,y) = 0$$

In the particular the case where $\nu(t,y) = \nu(t)$, straightforward computation leads to

$$Y_t(y) = e^{\frac{1}{2}\int_0^t \nu^2(s)ds}(y + \int_0^t e^{-\frac{1}{2}\int_0^s \nu^2(u)du}\nu(s)dB_s)\,.$$

Setting $f(s) = -\nu(s)\exp(-\frac{1}{2}\int_0^s \nu^2(u)du)$, one deduces that $Y_t(y) = \frac{y-m_t}{\sigma(t)}$, where $\sigma^2(t) = \int_t^\infty f^2(s)ds$ and $m_t =: \int_0^t f(s)dB_s$, and we recover the Gaussian example of Subsection 5.4.4.

9.3 Song's results

9.3.1 An example with same survival processes

We recall that, starting with a survival martingale process $\widetilde{G}_t(\theta)$, one can construct other survival martingale processes $G_t(\theta)$ admitting the same survival process (i.e., $\widetilde{G}_t(t) = G_t(t)$), in particular the same intensity. The construction is based on the general result obtained in Jeanblanc and Song [?]: for any supermartingale Z valued in $[0,1[$, with multiplicative decomposition $Ne^{-\Lambda}$, where Λ is continuous, the family

$$G_t(\theta) = 1 - (1 - Z_t)\exp\left(-\int_\theta^t \frac{Z_s}{1-Z_s} d\Lambda_s\right) \quad 0 < \theta \le t \le \infty,$$

is a martingale survival process (called the basic martingale survival process) which satisfies $G_t(t) = Z_t$ and, if N is continuous, $dG_t(\theta) = \frac{1-G_t(\theta)}{1-Z_t} e^{-\Lambda_t} dN_t$. In particular, the associated intensity process is Λ (we emphasize that the intensity process does not contain enough information about the conditional law).
We illustrate this construction in the Gaussian example presented in Section 5.4.4 where we set $Y_t = \frac{m_t - h(t)}{\sigma(t)}$. The multiplicative decomposition of the supermartingale $\widetilde{G}_t = P(\tau > t|\mathcal{F}_t^B)$ is $\widetilde{G}_t = N_t \exp\left(-\int_0^t \lambda_s ds\right)$ where

$$dN_t = N_t \frac{\varphi(Y_t)}{\sigma(t)\Phi(Y_t)} dm_t, \quad \lambda_t = \frac{h'(t)\,\varphi(Y_t)}{\sigma(t)\,\Phi(Y_t)}.$$

Using the fact that $\widetilde{G}_t(t) = \Phi(Y_t)$, one checks that the basic martingale survival process satisfies

$$dG_t(\theta) = (1 - G_t(\theta))\frac{f(t)\varphi(Y_t)}{\sigma(t)\Phi(-Y_t)} dB_t,\, t \ge \theta, \quad G_\theta(\theta) = \Phi(Y_\theta)$$

which provides a new example of martingale survival processes, with density process

$$g_t(\theta) = (1 - G_t) e^{-\int_\theta^t \frac{G_s}{1-G_s}\lambda_s ds} \frac{G_\theta \lambda_\theta}{1 - G_\theta},\, \theta \le t.$$

Other constructions of martingale survival processes having a given survival process can be found in [?], as well as constructions of local-martingales N such that $Ne^{-\Lambda}$ is valued in $[0,1]$ for a given increasing continuous process Λ.

Chapter 10

Last Passage Times

We now present the study of the law (and the conditional law) of some last passage times for diffusion processes. In this section, W is a standard Brownian motion and its natural filtration is $\mathbb{F}$. These random times have been studied in Jeanblanc and Rutkowski [68] as theoretical examples of default times, in Imkeller [61] as examples of insider private information and, in a pure mathematical point of view, in Pitman and Yor [100] and Salminen [103].

✓Profeta Roynette Yor, PLaten et Askhan

✓COMPUTE LAST PASSAGE TIME DENSITY

We show that, in a diffusion setup, the Doob-Meyer decomposition of the Azéma supermartingale may be computed explicitly for some random times τ.

10.1 Last Passage Time of a Transient Diffusion

Proposition 10.1.1 *Let X be a transient homogeneous diffusion such that $X_t \to +\infty$ when $t \to \infty$, and s a scale function such that $s(+\infty) = 0$ (hence, $s(x) < 0$ for $x \in \mathbb{R}$) and $\Lambda_y = \sup\{t : X_t = y\}$ the last time that X hits y. Then,*

$$\mathbb{P}_x(\Lambda_y > t|\mathcal{F}_t) = \frac{s(X_t)}{s(y)} \wedge 1 \,.$$

PROOF: We follow Pitman and Yor [100] and Yor [115, p.48], and use that under the hypotheses of the proposition, one can choose a scale function such that $s(x) < 0$ and $s(+\infty) = 0$ (see Sharpe [104]).

Observe that

$$\begin{aligned}
\mathbb{P}_x\big(\Lambda_y > t|\mathcal{F}_t\big) &= \mathbb{P}_x\Big(\inf_{u\geq t} X_u < y \,\Big|\, \mathcal{F}_t\Big) = \mathbb{P}_x\Big(\sup_{u\geq t}(-s(X_u)) > -s(y) \,\Big|\, \mathcal{F}_t\Big) \\
&= \mathbb{P}_{X_t}\Big(\sup_{u\geq 0}(-s(X_u)) > -s(y)\Big) = \frac{s(X_t)}{s(y)} \wedge 1,
\end{aligned}$$

where we have used the Markov property of X, and the fact that if M is a continuous local martingale with $M_0 = 1$, $M_t \geq 0$, and $\lim_{t\to\infty} M_t = 0$, then

$$\sup_{t\geq 0} M_t \overset{\text{law}}{=} \frac{1}{U} \,,$$

where U has a uniform law on $[0, 1]$ (see Exercise 1.5.3). □

The time Λ_y is honest: defining $\Lambda_y^t = \sup\{s \leq t : X_s = y\}$, one has $\Lambda_y = \Lambda_y^t$ on the set $\{\Lambda_y \leq t\}$.

Lemma 10.1.2 *The $\mathbb{F}^X$-predictable compensator A associated with the random time Λ_y is the process A defined as $A_t = -\frac{1}{2s(y)} L_t^{s(y)}(Y)$, where $L(Y)$ is the local time process of the continuous martingale $Y = s(X)$.*

Proof: From $x \wedge y = x - (x-y)^+$, Proposition 10.1.1 and Tanaka's formula, it follows that

$$\frac{s(X_t)}{s(y)} \wedge 1 = M_t + \frac{1}{2s(y)} L_t^{s(y)}(Y) = M_t + \frac{1}{s(y)} \ell_t^y(X)$$

where M is a martingale. The required result is then easily obtained. □

✓TANAKA FORMULA, DEFINE $p^{(m)}$

We deduce the law of the last passage time:

$$\begin{aligned}
\mathbb{P}_x(\Lambda_y > t) &= \left(\frac{s(x)}{s(y)} \wedge 1\right) + \frac{1}{s(y)} \mathbb{E}_x(\ell_t^y(X)) \\
&= \left(\frac{s(x)}{s(y)} \wedge 1\right) + \frac{1}{s(y)} \int_0^t du\, p_u^{(m)}(x,y)\,.
\end{aligned}$$

Hence, for $x < y$

$$\begin{aligned}
\mathbb{P}_x(\Lambda_y \in dt) &= -\frac{dt}{s(y)} p_t^{(m)}(x,y) = -\frac{dt}{s(y)m(y)} p_t(x,y) \\
&= -\frac{\sigma^2(y)s'(y)}{2s(y)} p_t(x,y)dt\,.
\end{aligned} \tag{10.1.1}$$

For $x > y$, we have to add a mass at point 0 equal to

$$1 - \left(\frac{s(x)}{s(y)} \wedge 1\right) = 1 - \frac{s(x)}{s(y)} = \mathbb{P}_x(T_y < \infty)\,.$$

Example 10.1.3 Last Passage Time for a Transient Bessel Process: For a Bessel process of dimension $\delta > 2$ and index ν (see [3M] Chapter 6), starting from 0,

$$\begin{aligned}
\mathbb{P}_0^\delta(\Lambda_a < t) &= \mathbb{P}_0^\delta(\inf_{u \geq t} R_u > a) = \mathbb{P}_0^\delta(\sup_{u \geq t} R_u^{-2\nu} < a^{-2\nu}) \\
&= \mathbb{P}_0^\delta\left(\frac{R_t^{-2\nu}}{U} < a^{-2\nu}\right) = \mathbb{P}_0^\delta(a^{2\nu} < U R_t^{2\nu}) = \mathbb{P}_0^\delta\left(\frac{a^2}{R_1^2 U^{1/\nu}} < t\right)\,.
\end{aligned}$$

Thus, the r.v. $\Lambda_a = \frac{a^2}{R_1^2 U^{1/\nu}}$ is distributed as $\frac{a^2}{2\gamma(\nu+1)\beta_{\nu,1}} \stackrel{\text{law}}{=} \frac{a^2}{2\gamma(\nu)}$ where $\gamma(\nu)$ is a gamma variable with parameter ν:

$$\mathbb{P}(\gamma(\nu) \in dt) = \mathbb{1}_{\{t \geq 0\}} \frac{t^{\nu-1}e^{-t}}{\Gamma(\nu)} dt\,.$$

Hence,

$$\mathbb{P}_0^\delta(\Lambda_a \in dt) = \mathbb{1}_{\{t \geq 0\}} \frac{1}{t\Gamma(\nu)} \left(\frac{a^2}{2t}\right)^\nu e^{-a^2/(2t)} dt\,. \tag{10.1.2}$$

We might also find this result directly from the general formula (10.1.1).

Proposition 10.1.4 *For H a positive predictable process*

$$\mathbb{E}_x(H_{\Lambda_y}|\Lambda_y = t) = \mathbb{E}_x(H_t|X_t = y)$$

and, for $y > x$,

$$\mathbb{E}_x(H_{\Lambda_y}) = \int_0^\infty \mathbb{E}_x(\Lambda_y \in dt)\, \mathbb{E}_x(H_t|X_t = y)\,.$$

In the case $x > y$,

$$\mathbb{E}_x(H_{\Lambda_y}) = H_0\left(1 - \frac{s(x)}{s(y)}\right) + \int_0^\infty \mathbb{E}_x(\Lambda_y \in dt)\, \mathbb{E}_x(H_t|X_t = y)\,.$$

Proof: We have shown in the previous Proposition 10.1.1 that

$$\mathbb{P}_x(\Lambda_y > t|\mathcal{F}_t) = \frac{s(X_t)}{s(y)} \wedge 1\,.$$

From Itô-Tanaka's formula

$$\frac{s(X_t)}{s(y)} \wedge 1 = \frac{s(x)}{s(y)} \wedge 1 + \int_0^t 1\!\!1_{\{X_u > y\}}\, d\frac{s(X_u)}{s(y)} - \frac{1}{2} L_t^{s(y)}(s(X))\,.$$

It follows, using Lemma 7.1.1 that

$$\begin{aligned}
\mathbb{E}_x(H_{\Lambda_x}) &= \frac{1}{2}\mathbb{E}_x\left(\int_0^\infty H_u\, d_u L_u^{s(y)}(s(X))\right) \\
&= \frac{1}{2}\mathbb{E}_x\left(\int_0^\infty \mathbb{E}_x(H_u|X_u = y)\, d_u L_u^{s(y)}(s(X))\right)\,.
\end{aligned}$$

Therefore, replacing H_u by $H_u g(u)$, we get

$$\mathbb{E}_x\left(H_{\Lambda_x} g(\Lambda_x)\right) = \frac{1}{2}\mathbb{E}_x\left(\int_0^\infty g(u)\, \mathbb{E}_x\left(H_u|X_u = y\right)\, d_u L_u^{s(y)}(s(X))\right)\,. \tag{10.1.3}$$

Consequently, from (10.1.3), we obtain

$$\begin{aligned}
\mathbb{P}_x\left(\Lambda_y \in du\right) &= \frac{1}{2} d_u \mathbb{E}_x\left(L_u^{s(y)}(s(X))\right) \\
\mathbb{E}_x\left(H_{\Lambda_y}|\Lambda_y = t\right) &= \mathbb{E}_x(H_t|X_t = y)\,.
\end{aligned}$$

□

Exercise 10.1.5 Let X be a drifted Brownian motion with positive drift ν and Λ_y^ν its last passage time at level y. Prove that

$$\mathbb{P}_x(\Lambda_y^{(\nu)} \in dt) = \frac{\nu}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(x - y + \nu t)^2\right) dt\,,$$

and

$$\mathbb{P}_x(\Lambda_y^{(\nu)} = 0) = \begin{cases} 1 - e^{-2\nu(x-y)}, & \text{for} \quad x > y \\ 0 & \text{for} \quad x < y\,. \end{cases}$$

Prove, using time inversion that, for $x = 0$,

$$\Lambda_y^{(\nu)} \overset{\text{law}}{=} \frac{1}{T_\nu^{(y)}}$$

where

$$T_a^{(b)} = \inf\{t \,:\, B_t + bt = a\}$$

See Madan et al. [92]. ◁

10.2 Last Passage Time Before Hitting a Level

Let $X_t = x + \sigma W_t$ where the initial value x is positive and σ is a positive constant. We consider, for $0 < a < x$ the last passage time at the level a before hitting the level 0, given as $g^a_{T_0}(X) = \sup\{t \leq T_0 \,:\, X_t = a\}$, where

$$T_0 = T_0(X) = \inf\{t \geq 0 \,:\, X_t = 0\}\,.$$

(In a financial setting, T_0 can be interpreted as the time of bankruptcy.) Then, setting $\alpha = (a-x)/\sigma$, $T_{-x/\sigma}(W) = \inf\{t \,:\, W_t = -x/\sigma\}$ and $d^\alpha_t(W) = \inf\{s \geq t \,:\, W_s = \alpha\}$

$$\mathbb{P}_x\big(g^a_{T_0}(X) \leq t|\mathcal{F}_t\big) = \mathbb{P}\big(d^\alpha_t(W) > T_{-x/\sigma}(W)|\mathcal{F}_t\big)$$

on the set $\{t < T_{-x/\sigma}(W)\}$. It is easy to prove that

$$\mathbb{P}\big(d^\alpha_t(W) < T_{-x/\sigma}(W)|\mathcal{F}_t\big) = \Psi(W_{t\wedge T_{-x/\sigma}(W)}, \alpha, -x/\sigma),$$

where the function $\Psi(\cdot, a, b) : \mathbb{R} \to \mathbb{R}$ equals, for $a > b$,

$$\Psi(y, a, b) = \mathbb{P}_y(T_a(W) > T_b(W)) = \begin{cases} (a-y)/(a-b) & \text{for} \quad b < y < a, \\ 1 & \text{for} \quad a < y, \\ 0 & \text{for} \quad y < b. \end{cases}$$

(See Proposition **??** for the computation of Ψ.) Consequently, on the set $\{T_0(X) > t\}$ we have

$$\mathbb{P}_x\big(g^a_{T_0}(X) \leq t|\mathcal{F}_t\big) = \frac{(\alpha - W_{t\wedge T_0})^+}{a/\sigma} = \frac{(\alpha - W_t)^+}{a/\sigma} = \frac{(a - X_t)^+}{a}. \tag{10.2.1}$$

As a consequence, applying Tanaka's formula, we obtain the following result.

Lemma 10.2.1 *Let $X_t = x + \sigma W_t$, where $\sigma > 0$. The $\mathbb{F}$-predictable compensator associated with the random time $g^a_{T_0(X)}$ is the process A defined as $A_t = \frac{1}{2\alpha} L^\alpha_{t\wedge T_{-x/\sigma}(W)}(W)$, where $L^\alpha(W)$ is the local time of the Brownian Motion W at level $\alpha = (a-x)/\sigma$.*

10.3 Last Passage Time Before Maturity

In this subsection, we study the last passage time at level a of a diffusion process X before the fixed horizon (maturity) T. We start with the case where $X = W$ is a Brownian motion starting from 0 and where the level a is null:

$$g_T = \sup\{t \leq T \,:\, W_t = 0\}\,.$$

Lemma 10.3.1 *The $\mathbb{F}$-predictable compensator associated with the random time g_T equals*

$$A_t = \sqrt{\frac{2}{\pi}} \int_0^{t\wedge T} \frac{dL_s}{\sqrt{T-s}},$$

where L is the local time at level 0 of the Brownian motion W.

Proof: It suffices to give the proof for $T = 1$, and we work with $t < 1$. Let G be a standard Gaussian variable. Then

$$\mathbb{P}\Big(\frac{a^2}{G^2} > 1 - t\Big) = \Phi\Big(\frac{|a|}{\sqrt{1-t}}\Big),$$

where $\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x \exp(-\frac{u^2}{2})du$. For $t < 1$, the set $\{g_1 \leq t\}$ is equal to $\{d_t > 1\}$. It follows (see [3M]) that

$$\mathbb{P}(g_1 \leq t|\mathcal{F}_t) = \Phi\left(\frac{|W_t|}{\sqrt{1-t}}\right).$$

Then, the Itô-Tanaka formula combined with the identity

$$x\Phi'(x) + \Phi''(x) = 0$$

leads to

$$\begin{aligned}\mathbb{P}(g_1 \le t|\mathcal{F}_t) &= \int_0^t \Phi'\left(\frac{|W_s|}{\sqrt{1-s}}\right) d\left(\frac{|W_s|}{\sqrt{1-s}}\right) + \frac{1}{2}\int_0^t \frac{ds}{1-s}\,\Phi''\left(\frac{|W_s|}{\sqrt{1-s}}\right)\\ &= \int_0^t \Phi'\left(\frac{|W_s|}{\sqrt{1-s}}\right) \frac{\operatorname{sgn}(W_s)}{\sqrt{1-s}}\,dW_s + \int_0^t \frac{dL_s}{\sqrt{1-s}}\,\Phi'\left(\frac{|W_s|}{\sqrt{1-s}}\right)\\ &= \int_0^t \Phi'\left(\frac{|W_s|}{\sqrt{1-s}}\right) \frac{\operatorname{sgn}(W_s)}{\sqrt{1-s}}\,dW_s + \sqrt{\frac{2}{\pi}}\int_0^t \frac{dL_s}{\sqrt{1-s}}\,.\end{aligned}$$

It follows that the $\mathbb{F}$-predictable compensator associated with g_1 is

$$A_t = \sqrt{\frac{2}{\pi}}\int_0^t \frac{dL_s}{\sqrt{1-s}},\ (t<1)\,.$$

□

These results can be extended to the last time before T when the Brownian motion reaches the level α, i.e., $g_T^\alpha = \sup\{t \le T\,:\, W_t = \alpha\}$, where we set $\sup(\emptyset) = T$. The predictable compensator associated with g_T^α is

$$A_t = \sqrt{\frac{2}{\pi}}\int_0^{t\wedge T} \frac{dL_s^\alpha}{\sqrt{T-s}},$$

where L^α is the local time of W at level α.

We now study the case where $X_t = x + \mu\, t + \sigma\, W_t$, with constant coefficients μ and $\sigma > 0$. Let

$$\begin{aligned} g_1^a(X) &= \sup\{t \le 1\,:\, X_t = a\}\\ &= \sup\{t \le 1\,:\, \nu t + W_t = \alpha\}\end{aligned}$$

where $\nu = \mu/\sigma$ and $\alpha = (a-x)/\sigma$. Setting

$$V_t = \alpha - \nu t - W_t = (a - X_t)/\sigma\,,$$

we obtain, using standard computations (see [3M])

$$\mathbb{P}(g_1^a(X) \le t|\mathcal{F}_t) = (1 - e^{\nu V_t} H(\nu, |V_t|, 1-t))\mathbb{1}_{\{T_0(V)\le t\}},$$

where

$$H(\nu, y, s) = e^{-\nu y}\mathcal{N}\left(\frac{\nu s - y}{\sqrt{s}}\right) + e^{\nu y}\mathcal{N}\left(\frac{-\nu s - y}{\sqrt{s}}\right)\,.$$

Using Itô's lemma, we obtain the decomposition of $1 - e^{\nu V_t} H(\nu, |V_t|, 1-t)$ as a semi-martingale $M_t + C_t$.

We note that C increases only on the set $\{t\,:\, X_t = a\}$. Indeed, setting $g_1^a(X) = g$, for any predictable process H, one has

$$\mathbb{E}(H_g) = \mathbb{E}\left(\int_0^\infty dC_s H_s\right)$$

hence, since $X_g = a$,

$$0 = \mathbb{E}(\mathbb{1}_{X_g \ne a}) = \mathbb{E}\left(\int_0^\infty dC_s \mathbb{1}_{X_s \ne a}\right)\,.$$

Therefore, $dC_t = \kappa_t dL_t^a(X)$ and, since L increases only at points such that $X_t = a$ (i.e., $V_t = 0$), one has

$$\kappa_t = H'_x(\nu, 0, 1-t)\,.$$

The martingale part is given by $dM_t = m_t dW_t$ where

$$m_t \;=\; e^{\nu V_t}\left(\nu H(\nu, |V_t|, 1-t) - \mathrm{sgn}(V_t)\, H'_x(\nu, |V_t|, 1-t)\right)\,.$$

Therefore, the predictable compensator associated with $g_1^a(X)$ is

$$\int_0^t \frac{H'_x(\nu, 0, 1-s)}{e^{\nu V_s} H(\nu, 0, 1-s)} dL_s^a\,.$$

Exercise 10.3.2 The aim of this exercise is to compute, for $t < T < 1$, the quantity $\mathbb{E}(h(W_T)\mathbb{1}_{\{T<g_1\}}|\mathcal{G}_t)$, which is the price of the claim $h(S_T)$ with barrier condition $\mathbb{1}_{\{T<g_1\}}$.

Prove that

$$\mathbb{E}(h(W_T)\mathbb{1}_{\{T<g_1\}}|\mathcal{F}_t) = \mathbb{E}(h(W_T)|\mathcal{F}_t) - \mathbb{E}\Big(h(W_T)\Phi\Big(\frac{|W_T|}{\sqrt{1-T}}\Big)\,\Big|\,\mathcal{F}_t\Big)\,,$$

where

$$\Phi(x) = \sqrt{\frac{2}{\pi}}\int_0^x \exp\left(-\frac{u^2}{2}\right) du\,.$$

Define $k(w) = h(w)\Phi(|w|/\sqrt{1-T})$. Prove that $\mathbb{E}\Big(k(W_T)\,\Big|\,\mathcal{F}_t\Big) = \widetilde{k}(t, W_t)$, where

$$\begin{aligned}\widetilde{k}(t,a) &= \mathbb{E}\Big(k(W_{T-t}+a)\Big)\\ &= \frac{1}{\sqrt{2\pi(T-t)}}\int_{\mathbb{R}} h(u)\Phi\Big(\frac{|u|}{\sqrt{1-T}}\Big)\exp\Big(-\frac{(u-a)^2}{2(T-t)}\Big)\,du.\end{aligned}$$

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10.4 Absolutely Continuous Compensator

From the preceding computations, the reader might think that the $\mathbb{F}$-predictable compensator is always singular w.r.t. the Lebesgue measure. This is not the case, as we show now. We are indebted to Michel Émery for this example.

Let W be a Brownian motion and let $\tau = \sup\{t \le 1 \,:\, W_1 - 2W_t = 0\}$, that is the last time before 1 when the Brownian motion is equal to half of its terminal value at time 1. Then,

$$\{\tau \le t\} = \Big\{\inf_{t\le s\le 1} 2W_s \ge W_1 \ge 0\Big\} \cup \Big\{\sup_{t\le s\le 1} 2W_s \le W_1 \le 0\Big\}\,.$$

▶ The quantity

$$\mathbb{P}(\tau \le t, W_1 \ge 0|\mathcal{F}_t) = \mathbb{P}\left(\inf_{t\le s\le 1} 2W_s \ge W_1 \ge 0|\mathcal{F}_t\right)$$

can be evaluated using the equalities

$$\begin{aligned}\Big\{\inf_{t\le s\le 1} W_s \ge \frac{W_1}{2} \ge 0\Big\} &= \Big\{\inf_{t\le s\le 1}(W_s - W_t) \ge \frac{W_1}{2} - W_t \ge -W_t\Big\}\\ &= \Big\{\inf_{0\le u\le 1-t}(\widetilde{W}_u) \ge \frac{\widetilde{W}_{1-t}}{2} - \frac{W_t}{2} \ge -W_t\Big\}\,,\end{aligned}$$

where $(\widetilde{W}_u = W_{t+u} - W_t, u \geq 0)$ is a Brownian motion independent of $\mathcal{F}_t$. It follows that

$$\mathbb{P}\left(\inf_{t\leq s\leq 1} W_s \geq \frac{W_1}{2} \geq 0 | \mathcal{F}_t\right) = \Psi(1-t, W_t)\,,$$

where

$$\begin{aligned}\Psi(s,x) &= \mathbb{P}\left(\inf_{0\leq u\leq s} \widetilde{W}_u \geq \frac{\widetilde{W}_s}{2} - \frac{x}{2} \geq -x\right) = \mathbb{P}\left(2M_s - W_s \leq \frac{x}{2}, W_s \leq \frac{x}{2}\right) \\ &= \mathbb{P}\left(2M_1 - W_1 \leq \frac{x}{2\sqrt{s}}, W_1 \leq \frac{x}{2\sqrt{s}}\right).\end{aligned}$$

► The same kind of computation leads to

$$\mathbb{P}\left(\sup_{t\leq s\leq 1} 2W_s \leq W_1 \leq 0 | \mathcal{F}_t\right) = \Psi(1-t, -W_t)\,.$$

► The quantity $\Psi(s,x)$ can now be computed from the joint law of the maximum and of the process at time 1; however, we prefer to use Pitman's theorem (see [3M]): let $\widetilde{U}$ be a r.v. uniformly distributed on $[-1,+1]$ independent of $R_1 := 2M_1 - W_1$, then

$$\begin{aligned}\mathbb{P}(2M_1 - W_1 \leq y, W_1 \leq y) &= \mathbb{P}(R_1 \leq y, \widetilde{U}R_1 \leq y) \\ &= \frac{1}{2}\int_{-1}^{1} \mathbb{P}(R_1 \leq y, uR_1 \leq y) du\,.\end{aligned}$$

For $y > 0$,

$$\frac{1}{2}\int_{-1}^{1} \mathbb{P}(R_1 \leq y, uR_1 \leq y) du = \frac{1}{2}\int_{-1}^{1} \mathbb{P}(R_1 \leq y) du = \mathbb{P}(R_1 \leq y)\,.$$

For $y < 0$

$$\int_{-1}^{1} \mathbb{P}(R_1 \leq y, uR_1 \leq y) du = 0\,.$$

Therefore

$$\mathbb{P}(\tau \leq t | \mathcal{F}_t) = \Psi(1-t, W_t) + \Psi(1-t, -W_t) = \rho\left(\frac{|W_t|}{\sqrt{1-t}}\right)$$

where

$$\rho(y) = \mathbb{P}(R_1 \leq y) = \sqrt{\frac{2}{\pi}}\int_0^y x^2 e^{-x^2/2} dx\,.$$

Then $Z_t = \mathbb{P}(\tau > t | \mathcal{F}_t) = 1 - \rho(\frac{|W_t|}{\sqrt{1-t}})$. We can now apply Tanaka's formula to the function ρ. Noting that $\rho'(0) = 0$, the contribution to the Doob-Meyer decomposition of Z of the local time of W at level 0 is 0. Furthermore, the increasing process A of the Doob-Meyer decomposition of Z is given by

$$\begin{aligned}dA_t &= \left(\frac{1}{2}\rho''\left(\frac{|W_t|}{\sqrt{1-t}}\right)\frac{1}{1-t} + \frac{1}{2}\rho'\left(\frac{|W_t|}{\sqrt{1-t}}\right)\frac{|W_t|}{\sqrt{(1-t)^3}}\right) dt \\ &= \frac{1}{1-t}\frac{|W_t|}{\sqrt{1-t}} e^{-W_t^2/2(1-t)} dt\,.\end{aligned}$$

We note that A may be obtained as the dual predictable projection on the Brownian filtration of the process $A_s^{(W_1)}, s \leq 1$, where $(A_s^{(x)}, s \leq 1)$ is the compensator of τ under the law of the Brownian bridge $\mathbb{P}_{0\to x}^{(1)}$.

Comment 10.4.1 Note that the random time τ presented in this subsection is not the end of a predictable set, hence, is not honest. However, $\mathbb{F}$-martingales are semi-martingales in the progressive enlarged filtration: it suffices to note that $\mathbb{F}$-martingales are semi-martingales in the filtration initially enlarged with W_1.

10.5 Time When the Supremum is Reached

Let W be a Brownian motion, $M_t = \sup_{s\leq t} W_s$ and let τ be the time when the supremum on the interval $[0,1]$ is reached, i.e.,

$$\tau = \inf\{t \leq 1 \,:\, W_t = M_1\} = \sup\{t \leq 1 \,:\, M_t - W_t = 0\}\,.$$

Let us denote by ζ the positive continuous semimartingale

$$\zeta_t = \frac{M_t - W_t}{\sqrt{1-t}},\ t < 1.$$

Let $F_t = \mathbb{P}(\tau \leq t|\mathcal{F}_t)$. Since $F_t = \Phi(\zeta_t)$, (where $\Phi(x) = \sqrt{\frac{2}{\pi}}\int_0^x \exp(-\frac{u^2}{2})du$, (see Exercise in Chapter 4 in [3M]) using Itô's formula, we obtain the canonical decomposition of F as follows:

$$\begin{aligned} F_t &= \int_0^t \Phi'(\zeta_u)\,d\zeta_u + \frac{1}{2}\int_0^t \Phi''(\zeta_u)\,\frac{du}{1-u} \\ &\overset{(i)}{=} -\int_0^t \Phi'(\zeta_u)\frac{dW_u}{\sqrt{1-u}} + \sqrt{\frac{2}{\pi}}\int_0^t \frac{dM_u}{\sqrt{1-u}} \overset{(ii)}{=} U_t + \tilde{F}_t, \end{aligned}$$

where $U_t = -\int_0^t \Phi'(\zeta_u)\dfrac{dW_u}{\sqrt{1-u}}$ is a martingale and $\widetilde{F}$ a predictable increasing process. To obtain (i), we have used that $x\Phi' + \Phi'' = 0$; to obtain (ii), we have used that $\Phi'(0) = \sqrt{2/\pi}$ and also that the process M increases only on the set

$$\{u \in [0,t] : M_u = W_u\} = \{u \in [0,t] : \zeta_u = 0\}.$$

10.6 Last Passage Times for Particular Martingales

We now study the Azéma supermartingale associated with the random time L, a last passage time or the end of a predictable set Γ, i.e.,

$$L(\omega) = \sup\{t \,:\, (t,\omega) \in \Gamma\}\,.$$

Proposition 10.6.1 *Let L be the end of a predictable set. Assume that all the $\mathbb{F}$-martingales are continuous and that L avoids the $\mathbb{F}$-stopping times. Then, there exists a continuous and nonnegative local martingale N, with $N_0 = 1$ and $\lim_{t\to\infty} N_t = 0$, such that:*

$$Z_t = \mathbb{P}\,(L > t \mid \mathcal{F}_t) = \frac{N_t}{\Sigma_t}$$

where $\Sigma_t = \sup_{s\leq t} N_s$. The Doob-Meyer decomposition of Z is

$$Z_t = m_t - A_t$$

and the following relations hold

$$\begin{aligned} N_t &= \exp\left(\int_0^t \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d\langle m\rangle_s}{Z_s^2}\right) \\ \Sigma_t &= \exp(A_t) \\ m_t &= 1 + \int_0^t \frac{dN_s}{\Sigma_s} = \mathbb{E}(\ln S_\infty|\mathcal{F}_t) \end{aligned}$$

PROOF: As recalled previously, the Doob-Meyer decomposition of Z reads $Z_t = m_t - A_t$ with m and A continuous, and dA_t is carried by $\{t : Z_t = 1\}$. Then, for $t < T_0 := \inf\{t : Z_t = 0\}$

$$-\ln Z_t = -\left(\int_0^t \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d\langle m\rangle_s}{Z_s^2}\right) + A_t$$

From Skorokhod's reflection lemma we deduce that

$$A_t = \sup_{u\le t}\left(\int_0^u \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^u \frac{d\langle m\rangle_s}{Z_s^2}\right)$$

Introducing the local martingale N defined by

$$N_t = \exp\left(\int_0^t \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^t \frac{d\langle m\rangle_s}{Z_s^2}\right),$$

it follows that

$$Z_t = \frac{N_t}{\Sigma_t}$$

and

$$\Sigma_t = \sup_{u\le t} N_u = \exp\left(\sup_{u\le t}\left(\int_0^u \frac{dm_s}{Z_s} - \frac{1}{2}\int_0^u \frac{d\langle m\rangle_s}{Z_s^2}\right)\right) = e^{A_t}$$

□

The three following exercises are from the work of Bentata and Yor [18].

Exercise 10.6.2 Let M be a positive martingale, such that $M_0 = 1$ and $\lim_{t\to\infty} M_t = 0$. Let $a \in [0,1[$ and define $G_a = \sup\{t : M_t = a\}$. Prove that

$$\mathbb{P}(G_a \le t|\mathcal{F}_t) = \left(1 - \frac{M_t}{a}\right)^+$$

Assume that, for every $t > 0$, the law of the r.v. M_t admits a density $(m_t(x), x \ge 0)$, and $(t,x) \to m_t(x)$ may be chosen continuous on $(0,\infty)^2$ and that $d\langle M\rangle_t = \sigma_t^2 dt$, and there exists a jointly continuous function $(t,x) \to \theta_t(x) = \mathbb{E}(\sigma_t^2|M_t = x)$ on $(0,\infty)^2$. Prove that

$$\mathbb{P}(G_a \in dt) = (1 - \frac{M_0}{a})\delta_0(dt) + 1\!\!1_{\{t>0\}}\frac{1}{2a}\theta_t(a)m_t(a)dt$$

`Hint:` Use Tanaka's formula to prove that the result is equivalent to $d_t\mathbb{E}(L_t^a(M)) = dt\theta_t(a)m_t(a)$ where L is the Tanaka-Meyer local time. ◁

Exercise 10.6.3 Let B be a Brownian motion and

$$\begin{aligned} T_a^{(\nu)} &= \inf\{t : B_t + \nu t = a\} \\ G_a^{(\nu)} &= \sup\{t : B_t + \nu t = a\} \end{aligned}$$

Prove that

$$(T_a^{(\nu)}, G_a^{(\nu)}) \stackrel{\text{law}}{=} \left(\frac{1}{G_\nu^{(a)}}, \frac{1}{T_\nu^{(a)}}\right)$$

Give the law of the pair $(T_a^{(\nu)}, G_a^{(\nu)})$. ◁

Exercise 10.6.4 Let X be a transient diffusion, such that

$$\begin{aligned}\mathbb{P}_x(T_0 < \infty) &= 0, x > 0\\ \mathbb{P}_x(\lim_{t\to\infty} X_t = \infty) &= 1, x > 0\end{aligned}$$

and note s the scale function satisfying $s(0^+) = -\infty, s(\infty) = 0$. Prove that for all $x, t > 0$,

$$\mathbb{P}_x(G_y \in dt) = \frac{-1}{2s(y)} p_t^{(m)}(x, y)dt$$

where $p^{(m)}$ is the density transition w.r.t. the speed measure m. ◁

Chapter 11

Solutions of some exercises

✓CHECK NUMBERS

11.1 Chapter 1

Exercise 1.1.4: Given the non right-continuous filtration $\mathbb{F}^0 = (\mathcal{F}_t^0)_{\mathbf{te0}}$, we define the smallest right-continuous filtration $\mathbb{F}$ containing $\mathbb{F}^0$ as follows: for any $t \geq 0$

$$\mathcal{F}_t := \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}^0.$$

Indeed, it is by definition right continuous and it satisfies $\mathcal{F}_t \supseteq \mathcal{F}_t^0$ for any t.

Exercise 1.1.7 Take $\mathcal{F}_t = \{\emptyset, \Omega\}$ for $t < 1$ and $\mathcal{F}_t = \sigma(X)$ for $t \geq 1$.

11.1.1 Predictable and optional σ-algebra

Exercise ?? Since we know that any measurable deterministic function is a predictable process, it suffices to provide an example of deterministic right-continuous function (recall that a process X is deterministic if, roughly speaking, it does not depend on the random parameter ω or, more exactly, if it is a stochastic process on $(\Omega, \{\Omega, \emptyset\})$). One example could be the following process X, defined, for $n \in \mathbb{N}$ and $t \geq 0$, as

$$(t, \omega) \mapsto X_t(\omega) := \sum_{n=0}^{\infty} n \mathbb{1}_{[n,n+1)}(t).$$

11.1.2 Semi-martingales

Exercise 1.2.7: It can be proved (for this we refer to the subsequent section "Poisson bridge", first exercise) that $(M_t := N_t - \lambda t)_t$ is an $\mathbb{F}^N$ martingale, where $\mathbb{F}^N$ is the natural filtration of N. In order to prove that the given decomposition is that of a semi-martingale, it remains, then, to prove that $(1-\theta)N_t + \theta\lambda t$ is an adapted finite variation process.
Recall that (cf. Definition 1.1.10.8 in [3M]) a càd process A is of finite variation if it is of finite variation on any compact set $[0, t]$. We say that it is of finite variation on $[0, t]$ if

$$V_A(t, \omega) = \sup \sum_{i=1}^{n} |A_{t_i}(\omega) - A_{t_{i-1}}(\omega)| = \int_0^t |dA_s(\omega)|$$

is a.s. finite, where the "sup" is taken over all possible partitions $(t_i)_{i=1,\ldots,n}$ of $[0, t]$.
The process $(1-\theta)N_t + \theta\lambda t$ is clearly adapted (notice that $\theta\lambda t$ is deterministics) and, furthermore,

it is of finite variation, since

$$\int_0^t |dN_s(\omega)| = \int_0^t dN_s(\omega) = N_t(\omega)$$

and $\mathbb{P}(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$, meaning that the explosion time is infinite.

The decomposition having a predictable finite variation process is the one corresponding to $\theta = 1$. The most important thing to notice here is that N is not predictable (this can be proved by contradiction, see e.g. Exercise 8.2.2.3 in [3M] or Liptser-Shiryaev II, Section 18.4).

Exercise 1.2.8: By definition of natural filtration, for every $t \geq 0$, H_t is adapted to $\mathcal{H}_t$, where $(\mathcal{H}_t)_t = \mathbb{H}$, so that $\{\tau \leq t\} \in \mathcal{H}_t$ and τ is an $\mathbb{H}$-stopping time.
If $\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t, \forall t \geq q0$, then $\{\tau \leq t\} \in \mathcal{G}_t$ and τ is also a $\mathbb{G}$-stopping time, for any filtration $\mathbb{F}$.

Exercise 1.2.9: It follows immediately by using the properties of conditional expectation: for $s \leq t$ and given that $\mathcal{F}_s \subseteq \mathcal{G}_t$, we have

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}\{\mathbb{E}[M_t|\mathcal{G}_s]|\mathcal{F}_s\} = \mathbb{E}[M_s|\mathcal{F}_s] = M_s,$$

if M is $\mathbb{F}$-adapted.

Analogously, we have, for $s \leq t$,

$$\mathbb{E}[\widehat{M_t}|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[M_t|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[M_s|\mathcal{F}_s] = \widehat{M_s}.$$

Exercise 1.2.10: We start by proving that, for $s \leq t$, $\mathbb{E}[X_t \mathbb{1}_A] = \mathbb{E}[X_s \mathbb{1}_A]$, where X is an $\mathbb{F}-$adapted martingale and $A \in \mathcal{G}_s$.
Because of the independence of $\mathbb{F}$ of $\widetilde{\mathbb{F}}$, we can consider elements $A \in \mathcal{G}_s$ of the form $A = A_1 A_2$, where $A_1 \in \mathcal{F}_s$ and $A_2 \in \widetilde{\mathcal{F}_s}$. We then have

$$\begin{aligned}
\mathbb{E}[X_t \mathbb{1}_A] &= \mathbb{E}[\mathbb{1}_{A_2}\mathbb{E}[X_t \mathbb{1}_{A_1}|\widetilde{\mathcal{F}_s}]] = \mathbb{E}[\mathbb{1}_{A_2}\mathbb{E}[X_t \mathbb{1}_{A_1}]] \\
&= \mathbb{E}[\mathbb{1}_{A_2}\mathbb{1}_{A_1}\mathbb{E}[X_t|\mathcal{F}_s]] = \mathbb{E}[\mathbb{1}_{A_2}\mathbb{1}_{A_1}X_s].
\end{aligned}$$

Because of the linearity of conditional expectation, the result holds true if we consider random variables of the form $\phi := \sum_{i=1}^n a_i \mathbb{1}_{A_1^i}\mathbb{1}_{A_2^i}, a_i \in \mathbb{R}$, too. It suffices, to conclude, to remark that any $\mathcal{G}_s-$measurable random variable is (increasing) limit of functions of the above form and to apply the Monotone Class Theorem (MCT).

For the second part of the exercise, we recall that the Martingale Representation Theorem, in the case of Brownian filtration, states that any $(\mathbb{P}, \mathbb{F})$-martingale M can be written as

$$M_t = M_0 + \int_0^t \xi_s dWs,$$

for some $\mathbb{F}-$predictable process ξ (such that the above stochastic integral is well defined). From Girsanov's Theorem, furthermore, we know that if the Radon-Nikodým density of $\mathbb{Q}$ with respect to $\mathbb{P}$, in the filtration $\mathbb{F}$, is Z, satisfying $Z_0 = 1$ and $dZ_t = Z_t \eta_t dW_t$ (η $\mathbb{F}-$adapted), then

$$W_t^* := W_t - \int_0^t \eta_s ds, \quad t \geq 0,$$

is a $(\mathbb{Q}, \mathbb{F})$-martingale. Because of the independence of $\mathbb{F}$ and $\widetilde{\mathbb{F}}$ under $\mathbb{Q}$, from the first part of the exercise it follows that W^* is a $(\mathbb{Q}, \mathbb{G})$-martingale, too. To conclude, we apply once more Girsanov theorem to pass, from measure $\mathbb{Q}$ to measure $\mathbb{P}$ in the filtration $\mathbb{G}$ and we immediately obtain the $(\mathbb{P}, \mathbb{G})$ semi-martingale decomposition of M.

(Ex. 1.1.13) If X and Y are continuous, then $\Delta[X, Y]_t = \Delta X_t \Delta Y_t = 0$ and the covariation process is continuous and equal to $< X, Y >$.

Let us recall a general result: if X is a stochastic process with independent, stationary increments (in French, "P.A.I.S."), satisfying $\mathbb{E}[|X_t|] < \infty, \forall t$ and $\mathbb{E}[|X_t^2|] < \infty, \forall t$, then

$$M_t := X_t - \mathbb{E}(X_t)$$

and

$$M_t^2 - \mathbb{E}(X_t^2)$$

are $\mathbb{F}^X-$martingales, where $\mathbb{F}^X$ denotes the natural filtration associated to X.

In particular, in the case of a Poisson process N with deterministic constant intensity λ, $M_t := N_t - \lambda t$ and $M_t^2 - \lambda t$ are $\mathbb{F}^N$ martingales. From the definition of predictable quadratic variation process of M we then find that $< M >_t = \lambda t$ (the deterministic process $(\lambda t)_t$ is predictable). For what concerns the quadratic variation of M, this is by definition the limit in probability of the sum of the square of the increments, i.e.,

$$[M, M]_t = \mathbb{P} - \lim \sum_{i=1}^{p(n)} (M_{t_i}^n - M_{t_{i-1}}^n)^2,$$

where $0 = t_0 \leq t_1^n \cdots \leq t_{p(n)}^n = t$ and $\sup_i(t_i^n - t_{i-1}^n)$ goes to zero. We then find $[M] = N$, since $(\Delta N)^2 = \Delta N$.

11.1.3 Definitions

By definition, we are looking for the predictable process ${}^{(p)}M$ that satisfies

$$\mathbb{E}(M_\tau \mathbb{1}_{\tau<\infty}|\mathcal{F}_{\tau_-}) = {}^{(p)}M_\tau \mathbb{1}_{\tau<\infty},$$

for any $\mathbb{F}-$predictable stopping time. A known result (see e.g. Dellacherie-Meyer, Vol. II, Ch. VI, Th. 32) states that given a càdlàg local martingale and a predictable stopping time τ we have

$$\mathbb{E}(M_\tau \mathbb{1}_{\tau<\infty}|\mathcal{F}_{\tau_-}) = M_{\tau_-} \mathbb{1}_{\tau<\infty}, \quad \text{a.s.}$$

and this gives us the desired result.

(Ex. 1.1.23) We consider $s \leq t$ and we compute the following conditional expectation

$$\mathbb{E}\left({}^{(o)}Y_t - \int_0^t {}^{(o)}X_u du \big| \mathcal{F}_s\right) \overset{\text{def}}{=} \mathbb{E}\left({}^{(o)}\left(\int_0^t X_u du\right) - \int_0^t {}^{(o)}X_u du \big| \mathcal{F}_s\right)$$
$$= \mathbb{E}\left(\mathbb{E}\left(\int_0^t X_u du|\mathcal{F}_t\right) - \int_0^t \mathbb{E}\left(X_u|\mathcal{F}_u\right) du \big| \mathcal{F}_s\right),$$

where we have used Theorem VI.7.10 in Rogers-Williams (1994), that states that if X is a bounded right-continuous process, then Z is indistinguishable from ${}^{(o)}X$ if and only if Z is an adapted right-continuous process such that $Z_t = \mathbb{E}[X_t|\mathcal{F}_t]$. We then find (notice that by setting the problem we have implicitly assumed that the above optional projections exist)

$$\mathbb{E}\left({}^{(o)}Y_t - \int_0^t {}^{(o)}X_u du \big| \mathcal{F}_s\right) = \mathbb{E}\left(\int_0^s X_u du|\mathcal{F}_s\right) - \int_0^s \mathbb{E}\left(X_u|\mathcal{F}_u\right) du$$
$$+\mathbb{E}\left(\mathbb{E}\left(\int_s^t X_u du|\mathcal{F}_t\right) - \int_s^t \mathbb{E}\left(X_u|\mathcal{F}_u\right) du \big| \mathcal{F}_s\right)$$
$$= {}^{(o)}Y_s - \int_0^s {}^{(o)}X_u du + \mathbb{E}\left(\int_s^t X_u du|\mathcal{F}_s\right) - \int_s^t \mathbb{E}\left(X_u|\mathcal{F}_s\right) du$$
$$= {}^{(o)}Y_s - \int_0^s {}^{(o)}X_u du.$$

(Ex. 1.1.24) By definition, given a predictable stopping time τ, we look for a process ${}^{(p)}(YX)$ such that

$$\mathbb{E}(Y_\tau X_\tau 1\!\!1_{\{\tau<\infty\}}|\mathcal{F}_{\tau-}) = {}^{(p)}(YX)1\!\!1_{\{\tau<\infty\}}.$$

By looking carefully at the definition of the $\sigma-$algebra $\mathcal{F}_{\tau-}$, it is clear that, for a predictable stopping time τ, the random variable $Y_\tau 1\!\!1_{\{\tau<\infty\}}$ is $\mathcal{F}_{\tau-}-$measurable and we have

$$\mathbb{E}(Y_\tau X_\tau 1\!\!1_{\{\tau<\infty\}}|\mathcal{F}_{\tau-}) = {}^{(p)}(X_\tau)Y_\tau 1\!\!1_{\{\tau<\infty\}},$$

the conclusion follows.

11.1.4 Examples

(Ex. 1.1.25) By using the results provided in the Example at page 10 of the notes, we have, applying Girsanov's theorem to pass from B_s to $B_s+\nu s$, for any s,

$$\begin{aligned}\mathbb{E}(f(B_s+\nu s)|\mathcal{F}_s^{|B|}) &= \frac{\mathbb{E}(f(B_s)e^{\nu B_s-\frac{\nu^2 s}{2}}|\mathcal{F}_s^{|B|})}{\mathbb{E}(e^{\nu B_s-\frac{\nu^2 s}{2}}|\mathcal{F}_s^{|B|})} = \frac{f(|B_s|)e^{\nu|B_s|}+f(-|B_s|)e^{-\nu|B_s|}}{2\cosh(\nu|B_s|)}\\ &= ?\end{aligned}$$

Once proven that the above projection exists, from Proposition 1.1.16 we know that $\left(\int_0^t f(B_s^{(\nu)})ds\right)^{(p)} = \int_0^t {}^{(p)}f(B_s^{(\nu)})ds$ and as predictable projection of the integrand we take the continuous process $\mathbb{E}(f(B_s^{(\nu)})|\mathcal{G}_s^{(\nu)})$.

(Ex. 1.1.26) We set $A_t := \int_0^t X_s d\alpha_s$, so that $dA_t = X_t d\alpha_t, t\geq 0$ and, given a positive $\mathbb{F}-$adapted process Y, we consider the following stochastic integral, looking for the integrable increasing $\mathbb{F}-$adapted process $(A_t)_t^{(p)}$ such that the second equality below holds true

$$\mathbb{E}\left(\int_0^\infty Y_s dA_s\right) = \mathbb{E}\left(\int_0^\infty Y_s X_s d\alpha_s\right) = \mathbb{E}\left(\int_0^\infty Y_s d{A_s}^{(p)}\right).$$

By hypothesis, $(\alpha_s, s\geq 0)$ is an increasing predictable process and so in order to find the integrable increasing $\mathbb{F}-$adapted process $(A_t)_t^{(p)}$ we only have to consider the predictable projection of X and, given the result in the previous Solution **??**,

$$\mathbb{E}\left(\int_0^\infty Y_s dA_s\right) = \mathbb{E}\left(\int_0^\infty Y_s X_s d\alpha_s\right) = \mathbb{E}\left(\int_0^\infty Y_s\, {}^{(p)}X_s d\alpha_s\right).$$

11.1.5 Some Exercices

Exercise 1.5.1: First of all we prove that effectively $\sup_{s\leq 1} B_s = \sup_{s\leq t} B_s \vee (\widehat{M}_{1-t}+B_t) = M_t\vee(\widehat{M}_{1-t}+B_t)$, by recalling that, given a Brownian motion B, the process $(B_{t+s}-B_s)_t =: (\hat{B}_t)_t$ denotes another Brownian motion. Then, exploiting the independence property of the increments of a Brownian motion and the measurability of M_t and B_t with respect to $\mathcal{F}_t$, we have

$$\mathbb{E}(f(M_1)|\mathcal{F}_t) = \mathbb{E}\left(f(M_t\vee(\widehat{M}_{1-t}+B_t))|\mathcal{F}_t\right) = \mathbb{E}\left(f(b\vee(\widehat{M}_{1-t}+a))\right)_{|a=B_t,b=M_t}.$$

The result is an immediate consequence of the fact that the random variable $\hat{M}_{1-t}$ has same law of $|\hat{B}_{1-t}|$.

Exercise 1.4.18: Let τ be the first jump time of a Poisson process and $\widetilde{M}$ the martingale $\widetilde{M}_t = N_t-\lambda t$. Then $Z_t = 1\!\!1_{t<\tau} = 1-1\!\!1_{\tau\leq t} = 1-A_t^{(p)}$ whereas the Doob-Meyer decomposition of Z is $Z_t = \widetilde{M}_{t\wedge\tau}-\lambda(t\wedge\tau]$

Exercise 1.5.2: As a first step we assume that φ is C^2. Then, from integration by parts and using the fact that B^* is increasing

$$(B_t^* - B_t)\varphi'(B^*t) = \int_0^t \varphi'(B_s^*)\, d(B_s^* - B_s) + \int_0^t (B_s^* - B_s)\varphi''(B_s^*)dB_s^* \,.$$

Now, we note that $\int_0^t (B_s^* - B_s)\varphi''(B_s^*)dB_s^* = 0$, since dB^*s is carried by $\{s : B_s^* = B_s\}$, and that $\int_0^t \varphi'(B_s^*)dB_s^* = \varphi(B_t^*) - \varphi(0)$. The result follows. The general case is obtained using the Monotone Class Theorem.

Exercise 1.5.3: (i) Let us consider the case $x < a$ and introduce $T_a := \inf\{t \geq 0 : M_t \geq a\}$. By applying Doob's optional sampling Theorem to the martingale M and to the finite stopping time $T_a \wedge t$, we find

$$\begin{aligned} x &= \mathbb{E}(M_{T_a \wedge t}) = a\mathbb{P}(T_a \leq t) + \mathbb{E}(M_t 1\!\!1_{\{T_a > t\}}) \\ &= a\mathbb{P}\left(\sup_{0 \leq s \leq t} M_s \geq qa\right) + \mathbb{E}\left(M_t 1\!\!1_{\{T_a > t\}}\right). \end{aligned} \tag{11.1.1}$$

By letting t go to infinity, recalling that, by hypothesis, $\lim_{t\to\infty} M_t = 0$ and thanks to the dominated convergence theorem, we finally find

$$\mathbb{P}(\sup_{0 \leq s \leq +\infty} M_s \geq a) = \mathbb{P}(\sup_t M_t \geq a) = \frac{x}{a}.$$

In the case when $x \leq a$, evidently $\mathbb{P}(T_a \leq t) = 1$ and the result follows. Furthermore, $\mathbb{P}(\frac{x}{U} \geq a) = \mathbb{P}(U \leq \frac{x}{a}) = \left(\frac{x}{a}\right) \wedge 1$.

(ii) We consider Equation (11.1.1) in the case when $x = 1$, namely

$$1 = \mathbb{E}(M_{T_a \wedge t}) = a\mathbb{P}\left(\sup_{0 \leq s \leq t} M_s \geq qa\right) + \mathbb{E}\left(M_t 1\!\!1_{\{T_a > t\}}\right)$$

and we let t go to infinity, obtaining

$$1 = a\mathbb{P}\left(\sup_t M_t \geq qa\right) + \mathbb{E}\left(M_\infty 1\!\!1_{\{T_a > +\infty\}}\right).$$

If we know that $\sup M_t \overset{law}{=} \frac{x}{U} = \frac{1}{U}$, choosing $a > 1$ we have

$$1 = a\frac{1}{a} + \mathbb{E}\left(M_\infty 1\!\!1_{\{T_a > +\infty\}}\right),$$

meaning that $\mathbb{E}\left(M_\infty 1\!\!1_{\{T_a > +\infty\}}\right) = 0$. Now, $\mathbb{P}(T_a > +\infty) = \mathbb{P}(\sup_t M_t < a) = 1 - \left(\frac{1}{a} \wedge 1\right) = 1 - \frac{1}{a}$ and for $a \to +\infty$ we have $\mathbb{P}(T_a > +\infty) \to 1$ and it follows that $\mathbb{E}(M_\infty) = 0$ and, being M positive, $M_\infty = 0$ a.s.

Exercise 1.5.5: Let us compute the conditional expectation $\mathbb{E}(M_t|\mathcal{F}_s)$, for $s \leq t$, where $(M_t)_t := \left(\mathbb{E}(\int_0^t a_u du|\mathcal{F}_t) - \int_0^t \mathbb{E}(a_u|\mathcal{F}_u)du\right)_t$.

$$\begin{aligned} \mathbb{E}(M_t|\mathcal{F}_s) &= \mathbb{E}\left(\mathbb{E}(\int_0^t a_u du|\mathcal{F}_t) - \int_0^t \mathbb{E}(a_u|\mathcal{F}_u)du|\mathcal{F}_s\right) \\ &= \mathbb{E}(\int_0^t a_u du|\mathcal{F}_s) - \mathbb{E}\left(\int_0^s \mathbb{E}(a_u|\mathcal{F}_u)du + \int_s^t \mathbb{E}(a_u|\mathcal{F}_u)du|\mathcal{F}_s\right) \\ &= \mathbb{E}(\int_0^s a_u du|\mathcal{F}_s) + \mathbb{E}(\int_s^t a_u du|\mathcal{F}_s) - \int_0^s \mathbb{E}(a_u|\mathcal{F}_u)du - \mathbb{E}\left(\int_s^t \mathbb{E}(a_u|\mathcal{F}_u)du|\mathcal{F}_s\right) \\ &= M_s + \mathbb{E}(\int_s^t a_u du|\mathcal{F}_s) - \int_s^t \mathbb{E}(a_u|\mathcal{F}_s)du = M_s. \end{aligned}$$

11.2 Chapter 2

11.3 Chapter 3

11.3.1 Definition

Exercise 3.1.2:

By definition (see e.g. [3M], Definition 1.4.1.1) a continuous process X is said to be a Brownian motion, if, between the others, one of the following equivalent properties is satisfied: either the processes $(X_t)_{t\geq 0}$ and $(X_t^2 - t)_{t\geq 0}$ are $\mathbb{F}^X-$local martingales, or, for any $\lambda \in \mathbb{R}$, $\Big(\exp(\lambda X_t - \frac{\lambda^2}{2}t)\Big)_{t\geq 0}$ is an $\mathbb{F}^X-$local martingale.
Since immersion property holds between $\mathbb{F}$ and $\mathbb{G}$, the result is immediate.

Exercise ??: Because of property $(\mathcal{H}_2)$ in Proposition 2.1.1, given $A \in \mathcal{G}_t \cap \mathcal{F}_\infty$, we have $\mathbb{1}_A = \mathbb{E}(\mathbb{1}_A|\mathcal{F}_\infty) = \mathbb{E}(\mathbb{1}_A|\mathcal{F}_t)$, meaning that $A \in \mathcal{F}_t$. Conversely, if $B \in \mathcal{F}_t$, it also holds $B \in \mathcal{G}_t$ and $B \in \mathcal{F}_\infty$ and we have equality between $\mathcal{F}_t$ and $\mathcal{G}_t \cap \mathcal{F}_\infty$.

11.3.2 Change of probability

Exercise 3.1.9: the first part of the exercise corresponds to Proposition 2 in Jeulin-Yor).
First of all let us recall that Girsanov's theorem tranforms $(\mathbb{Q},\mathbb{F})$ (resp. $(\mathbb{Q},\mathbb{G})$) semi-martingales into $(\mathbb{P},\mathbb{F})$ (resp. $(\mathbb{P},\mathbb{G})$) semi-martingales. Then, given a $(\mathbb{Q},\mathbb{F})$ semi-martingale X, thank to Girsanov's theorem it becomes a $(\mathbb{P},\mathbb{F})$ semi-martingale, but $(\mathcal{H})$ hypothesis holds between $\mathbb{F}$ and $\mathbb{G}$ under $\mathbb{P}$ so that, *a fortiori* any $(\mathbb{P},\mathbb{F})$ semi-martingale remains a $(\mathbb{P},\mathbb{G})$ semi-martingale and it remains to apply once more Girsanov's theorem to go back to $\mathbb{Q}$.

More precisely, given a $(\mathbb{Q},\mathbb{F})-$martingale X and noticing that

$$\frac{d\mathbb{P}}{d\mathbb{Q}}_{|\mathcal{F}_t} = \frac{1}{\Lambda_t}$$

we know, thanks to Girsanov's theorem (in its continuous version), that

$$\widetilde{X}_t := X_t - \int_0^t \Lambda_s d < X, \frac{1}{\Lambda} >_s, \quad t \geq 0,$$

is a $(\mathbb{P},\mathbb{F})$-(local) martingale, that remains a $(\mathbb{P},\mathbb{G})$-(local) martingale, since $(\mathcal{H})$ hypothesis holds under $\mathbb{P}$ between $\mathbb{F}$ and $\mathbb{G}$. We then apply once more Girsanov's theorem to pass from $\mathbb{P}$ to $\mathbb{Q}$ under $\mathbb{G}$ by means of Λ, so that we obtain the following $(\mathbb{Q},\mathbb{G})$-(local) martingale, for any t, in the continuous filtration case,

$$\bar{X}_t := \widetilde{X}_t - \int_0^t \frac{1}{L_s} d < \widetilde{X}, L >_s = X_t - \int_0^t \Lambda_s d < X, \frac{1}{\Lambda} >_s - \int_0^t \frac{1}{L_s} d < X, L >_s .$$

In the discontinuous case it is necessary to use the Girsanov's theorem in its more general form. For this we refer to Theorem 3 in Jeulin-Yor.

Exercise 3.1.13 First of all let us notice that if L is independent of $\mathcal{F}_\infty$, then immersion holds between $\mathbb{F}$ and $\mathbb{F}^{(L)}$. We will show that independence is not only a sufficient, but also a necessary condition for immersion property to hold. We will exploit property $(\mathcal{H}_1)$ in Proposition 2.1.1., namely the fact that $(\mathcal{H})$ hypothesis is equivalent, for any $t \geq 0$, to the conditional independence of $\mathcal{F}_t^{(L)}$ and $\mathcal{F}_\infty$ given $\mathcal{F}_t$. Let us, then, consider a random variable in $\mathcal{F}_t^{(L)}$ of the form $F_t h(L)$, with $F_t \in \mathcal{F}_t$ and a random variable $F_\infty \in \mathcal{F}_\infty$. Immersion property is equivalent, then, to (we suppose that the random variable that we have considered satisfy suitable integrability conditions, so that the conditional expectation below is well defined)

$$\mathbb{E}(F_t h(L) F_\infty|\mathcal{F}_t) \overset{(\mathcal{H})}{=} \mathbb{E}(F_t h(L)|\mathcal{F}_t)\mathbb{E}(F_\infty|\mathcal{F}_t),$$

for any $t \geq 0$.
In particular, taking a constant random variable F_t and $t = 0$ we find

$$\mathbb{E}(h(L)F_\infty) = \mathbb{E}(h(L))\mathbb{E}(F_\infty)$$

and we find that immersion property is equivalent to the independence of random variables of the form $h(L) \in \mathcal{F}_t^{(L)}$ and $F_\infty \in \mathcal{F}_\infty$. As usual, an application of the Monotone Class Theorem allows us to conclude the proof.

Exercise 3.1.14 To construct an example it suffices to recall Example 1.1.7, namely we define $\mathcal{G}_t := \mathcal{F}_\infty$, $\forall\, t \geq 0$. In this particular case, only constant $\mathbb{F}-$martingales remain $\mathbb{G}-$martingales.

Exercise 3.1.16 Immersion holds if and only if

$$\mathbb{E}(h(\tau)|\mathcal{G}_t) = \mathbb{E}(h(\tau)|\mathcal{G}_\infty)$$

Since $\mathcal{G}_\infty = \mathcal{F}_\infty \vee \sigma(\tau)$, this condition reduces to $\mathbb{E}(h(\tau)|\mathcal{G}_t) = h(\tau)$. In particular $\mathbb{E}(h(\tau)) = h(\tau)$, hence τ is constant.

11.4 Chapter 4

Exercise 5.2.5 Note that the result is obvious from the decomposition theorem: indeed taking expectation w.r.t. $\mathcal{F}_t$ of the two sides of

$$\widetilde{X}_t = X_t - \int_0^t \frac{d\langle p.(L), X\rangle_s}{p_{s-}(L)}$$

leads to

$$\mathbb{E}(\widetilde{X}_t|\mathcal{F}_t) = X_t - \mathbb{E}(\int_0^t \frac{d\langle p.(L), X\rangle_s}{p_{s-}(L)}|\mathcal{F}_t),$$

and $\mathbb{E}(\widetilde{X}_t|\mathcal{F}_t)$ is an $\mathbb{F}$-martingale.

Our aim is to check it directly. Writing $dp_t(u) = p_t(u)\sigma_t(u)dW_t$ and $dX_t = x_t dW_t$, we note that $\mathbb{P}(L \in \mathbb{R}|\mathcal{F}_s) = \int_\mathbb{R} p_s(u)\nu(du) = 1$ implies that

$$\int_\mathbb{R} p_s(u)\nu(du) = \int_\mathbb{R} p_0(u)\nu(du) + \int_0^t dW_s \int_\mathbb{R} \sigma_s(u)p_s(u)\nu(du) = 1 + \int_0^t dW_s \int_\mathbb{R} \sigma_s(u)p_s(u)\nu(du)$$

hence $\int_\mathbb{R} \sigma_s(u)p_s(u)\nu(du) = 0$. The process $\mathbb{E}(\int_0^t \frac{d\langle p.(L),X\rangle_s}{p_{s-}(L)}|\mathcal{F}_t)$ is equal to a martingale plus $\int_0^t \mathbb{E}(\sigma_s(L)x_s|\mathcal{F}_s)ds = \int_0^t ds\, x_s \int_\mathbb{R} \sigma_s(u)p_s(u)\nu(du) = 0$.

11.5 Chapter 6

Exercise 7.2.6 Let $\{Z_t > 0\} =: C_t$. Then $\mathbb{P}(C_t^c \cap \{\tau > t\}) = 0$.

Exercise?? See Section 2.2.2. **Exercise3.2.5** From the definition, immersion is equivalent to $\mathbb{E}(h(\tau)|\mathcal{G}_t) = h(\tau)$. For $t = 0$, one gets the result.

Exercise 7.4.6 From the key lemma,

$$I_t := \mathbb{E}\left(\int_0^{\tau\wedge T} G_s ds|\mathcal{G}_t\right) = 1\!\!1_{t<\tau}\frac{1}{G_t}\mathbb{E}(1\!\!1_{t<\tau}\int_0^{\tau\wedge T} G_s ds|\mathcal{F}_t) + 1\!\!1_{\tau\leq t}\int_0^\tau G_s ds$$

and from the second part of key lemma

$$\begin{aligned} I &:= \mathbb{E}(\mathbb{1}_{t<\tau}\int_0^{\tau\wedge T} G_s ds|\mathcal{F}_t) = \mathbb{E}(\int_t^\infty dF_u \int_0^{u\wedge T} G_s ds|\mathcal{F}_t) \\ &= \mathbb{E}(\int_t^T dF_u \int_0^u G_s ds|\mathcal{F}_t) + \mathbb{E}(\int_T^\infty dF_u \int_0^T G_s ds|\mathcal{F}_t) \\ &= \mathbb{E}(\int_t^T dF_u \int_0^u G_s ds|\mathcal{F}_t) + \mathbb{E}(G_T \int_0^T G_s ds|\mathcal{F}_t) \end{aligned}$$

From integration by parts formula, setting $\zeta_t = \int_0^t G_s ds$

$$G_T\zeta_T = G_t\zeta_t + \int_t^T \zeta_s dG_s + \int_t^T G_s d\zeta_s$$

hence

$$I = G_t\zeta_t + \mathbb{E}(\int_t^T G_s^2 ds|\mathcal{F}_t) = G_t\zeta_t - \int_0^t G_s^2 ds + m_t$$

and

$$\begin{aligned} I_t &:= \mathbb{E}(I|\mathcal{G}_t) = \mathbb{1}_{t<\tau}\left(\int_0^t G_s ds + \frac{1}{G_t}\left(m_t - \int_0^t G_s^2 ds\right)\right) + \mathbb{1}_{\tau\le t}\int_0^\tau G_u du \\ &= i_t\mathbb{1}_{t<\tau} + \mathbb{1}_{\tau\le t}\int_0^\tau G_u du = i_t\mathbb{1}_{t<\tau} + \int_0^t dH_s \int_0^s G_u du \end{aligned}$$

Differentiating this expression leads to

$$\begin{aligned} dI_t &= (\int_0^t G_s ds - i_t)dH_t + (1-H_t)\frac{1}{G_t^2}\left(m_t - \int_0^t G_s^2 ds\right)\left(-dZ_t + dA_t + \frac{1}{G_t}d\langle Z\rangle_t\right) \\ &\quad +(1-H_t)\left(\frac{1}{G_t^2}d\langle m,G\rangle_t + \frac{1}{G_t}dm_t - \frac{1}{G_t^2}(m_t - \int_0^t G_s^2 ds)d\langle m,G\rangle_t\right) \end{aligned}$$

After some simplifications, and using the fact that $dM_t = dH_t - (1-H_t)\frac{dA_t}{G_t}$

$$dI_t = \frac{1}{G_t}(m_t - \int_0^t G_s^2 ds)dM_t + (1-H_t)\frac{1}{G_t}(dm_t - \frac{1}{G_t}d\langle m,Z\rangle_t) - (1-H_t)\frac{1}{G_t^2}(m_t - \int_0^t G_s^2 ds)(dZ_t - \frac{1}{G_t}d\langle Z\rangle_t)$$

Exercise 3.2.7 Suppose that

$$P(\tau \le t|\mathcal{F}_\infty) = 1 - e^{-\Gamma_t}$$

where Γ is an arbitrary continuous strictly increasing $\mathbb{F}$-adapted process. Let us set $\Theta := \Gamma_\tau$. Then

$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

where C is the right inverse of Γ, so that $\Gamma_{C_t} = t$. Therefore

$$P(\Theta > u|\mathcal{F}_\infty) = e^{-\Gamma_{C_u}} = e^{-u}.$$

We have thus established the required properties, namely, the probability law of Θ and its independence of the σ-field $\mathcal{F}_\infty$. Furthermore, $\tau = \inf\{t : \Gamma_t > \Gamma_\tau\} = \inf\{t : \Gamma_t > \Theta\}$.

11.6 Chapter 7

Exercise 8.3.4 We give the proof in the case where $\mathbb{F}$ is a Brownian filtration. Then, for any θ, there exists two predictable processes $\gamma(\theta)$ and x such that

$$\begin{aligned} dX_t &= x_t dW_t \\ d_t p_t(\theta) &= p_t(\theta)\gamma_t(\theta)dW_t \end{aligned}$$

and $d\langle X, p(\theta)\rangle_t = x_t p_t(\theta)\gamma_t(\theta)dt$. It follows that $\int_0^t \frac{d\langle X,p(\theta)\rangle_s}{p_{s^-}(\theta)}\Big|_{\theta=\tau} = \int_0^t x_s\gamma_s(\tau)ds$. We write

$$\mathbb{E}\left(\int_0^t \frac{d\langle X, p(\theta)\rangle_s}{p_{s^-}(\theta)}|_{\theta=\tau}|\mathcal{G}_t\right) = \mathbb{E}\left(\int_0^{t\wedge\tau} x_s\gamma_s(\theta)ds|_{\theta=\tau}|\mathcal{G}_t\right) + \int_{t\wedge\tau}^t \frac{d\langle X, p(\theta)\rangle_s}{p_{s^-}(\theta)}$$

From Exercise 1.5.5,

$$\mathbb{E}\left(\int_0^{t\wedge\tau} x_s\gamma_s(\tau)ds|\mathcal{G}_t\right) = m_t + \int_0^t x_s\mathbb{E}(\mathbb{1}_{s<\tau}\gamma_s(\tau)|\mathcal{G}_s)ds$$

where m is a $\mathbb{G}$-martingale. From key lemma

$$\begin{aligned}\mathbb{E}(\mathbb{1}_{s<\tau}\gamma_s(\tau)|\mathcal{G}_s) &= \mathbb{1}_{s<\tau}\frac{1}{G_s}\mathbb{E}(\mathbb{1}_{<\tau}\gamma_s(\tau)|\mathcal{F}_s) = \mathbb{1}_{s<\tau}\frac{1}{G_s}\mathbb{E}(\int_s^\infty \gamma_s(u)p_s(u)\nu(du)|\mathcal{F}_s)\\ &= \mathbb{1}_{s<\tau}\frac{1}{G_s}\int_s^\infty \gamma_s(u)p_s(u)\nu(du)\end{aligned}$$

Now, we compute $d\langle X, G\rangle$. In order to compute this bracket, one needs the martingale part of G From $G_t = \int_t^\infty p_t(u)\nu(du)$, one deduces that

$$dG_t = \left(\int_t^\infty \gamma_t(u)p_t(u)\nu(du)\right)dW_t - p_t(t)\nu(dt)$$

The last property can be obtained from Itô-Kunita-Wentcell (this can also be checked by hands: indeed it is straightforward to check that $G_t + \int_0^t p_s(u)\nu(du)$ is a martingale). It follows that $d\langle X, G\rangle_t = x_t(\int_t^\infty \gamma_t(u)p_t(u)\nu(du))dt$, hence the result.

Bibliography

[1] A. Aksamit. *Random times, enlargement of filtration and arbitrages.* Thèse de doctorat, Univ. Evry, 2014.

[2] A. Aksamit, T. Choulli, J. Deng, and M. Jeanblanc. Arbitrages in a progressive enlargement setting. *Arbitrage, Credit and Informational Risks, Peking University Series in Mathematics*, 6:55–88, 2014.

[3] A. Aksamit, T. Choulli, J. Deng, and M. Jeanblanc. Decomposition of random times and their classification. 2014.

[4] A. Aksamit, T. Choulli, J. Deng, and M.. Jeanblanc. Non-arbitrage under a class of honest times. 2014.

[5] J. Amendinger. *Initial enlargement of filtrations and additional information in financial markets.* PhD thesis, Technischen Universität Berlin, 1999.

[6] J. Amendinger. Martingale representation theorems for initially enlarged filtrations. *Stochastic Processes and their Appl.*, 89:101–116, 2000.

[7] J. Amendinger, P. Imkeller, and M. Schweizer. Additional logarithmic utility of an insider. *Stochastic Processes and their Appl.*, 75:263–286, 1998.

[8] S. Ankirchner. *Information and Semimartingales.* PhD thesis, Humboldt Universität Berlin, 2005.

[9] S. Ankirchner, S. Dereich, and P. Imkeller. Enlargement of filtrations and continuous Girsanov-type embeddings. In C. Donati-Martin, M. Emery, A. Rouault, and Ch. Stricker, editors, *Séminaire de Probabilités XL*, volume 1899 of *Lecture Notes in Mathematics.* Springer-Verlag, 2007.

[10] T. Aven. A theorem for determining the compensator of a counting process. *Scandinavian Journal of Statistics*, 12:69–72, 1985.

[11] J. Azéma. Quelques applications de la théorie générale des processus, I. *Invent. Math.*, 18:293–336, 1972.

[12] Acciaio B., C. Fontana, and Kardaras C. Arbitrage of the first kind and filtration enlargements in semimartingale financial models. *arxiv.org/pdf/1401.7198.pdf*, 2014.

[13] P. Bank and D. Baum. Hedging and portfolio optimization in financial markets with a large trader. *Mathematical Finance*, 14:1Ũ18, 2004.

[14] M.T. Barlow. Study of filtration expanded to include an honest time. *Z. Wahr. Verw. Gebiete*, 44:307–323, 1978.

[15] M.T. Barlow. Study of filtration expanded to include an honest time. *Z. Wahr. Verw. Gebiete*, 44:307–323, 1978.

[16] F. Baudoin. Modeling anticipations on financial markets. In *Paris-Princeton Lecture on mathematical Finance 2002*, volume 1814 of *Lecture Notes in Mathematics*, pages 43–92. Springer-Verlag, 2003.

[17] M. Bedini. *Information on a Default Time: Brownian Bridges on Stochastic Intervals and Enlargement of Filtrations.* Thèse de doctorat, Jena University, 2012.

[18] A. Bentata and M. Yor. From Black-Scholes and Dupire formulae to last passage times of local martingales. *Preprint*, 2008.

[19] F. Biagini, Y. Lee, and T. Rheinländer. Hedging survivor bonds with mortality-linked securities. *Preprint*, 2010.

[20] T.R. Bielecki, M. Jeanblanc, and M. Rutkowski. Stochastic methods in credit risk modelling, valuation and hedging. In M. Frittelli and W. Rungaldier, editors, *CIME-EMS Summer School on Stochastic Methods in Finance, Bressanone*, volume 1856 of *Lecture Notes in Mathematics.* Springer, 2004.

[21] T.R. Bielecki, M. Jeanblanc, and M. Rutkowski. Market completeness under constrained trading. In A.N. Shiryaev, M.R. Grossinho, P.E. Oliveira, and M.L. Esquivel, editors, *Stochastic Finance, Proceedings of International Lisbon Conference*, pages 83–107. Springer, 2005.

[22] T.R. Bielecki, M. Jeanblanc, and M. Rutkowski. Completeness of a reduced-form credit risk model with discontinuous asset prices. *Stochastic Models*, 22:661–687, 2006.

[23] T.R. Bielecki and M. Rutkowski. *Credit risk: Modelling Valuation and Hedging.* Springer Verlag, Berlin, 2001.

[24] P. Brémaud. *Point Processes and Queues: Martingale Dynamics.* Springer Verlag, Berlin, 1981.

[25] P. Brémaud and M. Yor. Changes of filtration and of probability measures. *Z. Wahr. Verw. Gebiete*, 45:269–295, 1978.

[26] G. Callegaro, M. Jeanblanc, and B. Zargari. Carthagian enlargement of filtrations. *ESAIM: Probability and Statistics.*, 17:550 – 566, 2013.

[27] R. Carmona. Emissions option pricing. *slides, Heidelberg*, 2010.

[28] M. Chesney, M. Jeanblanc, and M. Yor. *Mathematical Methods in Financial Markets.* Springer, 2009.

[29] C.S. Chou and P-A. Meyer. Sur la représentation des martingales comme intégrales stochastiques dans les processus ponctuels. In P-A. Meyer, editor, *Séminaire de Probabilités IX*, volume 1557 of *Lecture Notes in Maths.*, pages 226–236. Springer-Verlag, 1975.

[30] P. Collin-Dufresne and J.-N. Hugonnier. On the pricing and hedging of contingent claims in the presence of extraneous risks. *SPA*, 117:742 – 765, 2007.

[31] J.M. Corcuera, P. Imkeller, A. Kohatsu-Higa, and D. Nualart. Additional utility of insiders with imperfect dynamical information. *Finance and Stochastics*, 8:437–450, 2004.

[32] M.H.A. Davis. Option Pricing in Incomplete Markets. In M.A.H. Dempster and S.R. Pliska, editors, *Mathematics of Derivative Securities*, Publication of the Newton Institute, pages 216–227. Cambridge University Press, 1997.

[33] C. Dellacherie. Un exemple de la théorie générale des processus. In P-A. Meyer, editor, *Séminaire de Probabilités IV*, volume 124 of *Lecture Notes in Mathematics*, pages 60–70. Springer-Verlag, 1970.

[34] C. Dellacherie. *Capacités et processus stochastiques*, volume 67 of *Ergebnisse.* Springer, 1972.

[35] C. Dellacherie, B. Maisonneuve, and P-A. Meyer. *Probabilités et Potentiel, chapitres XVII-XXIV, Processus de Markov (fin). Compléments de calcul stochastique.* Hermann, Paris, 1992.

[36] C. Dellacherie and P-A. Meyer. *Probabilités et Potentiel, chapitres I-IV.* Hermann, Paris, 1975. English translation: Probabilities and Potentiel, chapters I-IV, North-Holland, (1982).

[37] C. Dellacherie and P-A. Meyer. A propos du travail de Yor sur les grossissements des tribus. In C. Dellacherie, P-A. Meyer, and M. Weil, editors, *Séminaire de Probabilités XII*, volume 649 of *Lecture Notes in Mathematics*, pages 69–78. Springer-Verlag, 1978.

[38] C. Dellacherie and P-A. Meyer. *Probabilités et Potentiel, chapitres V-VIII.* Hermann, Paris, 1980. English translation : Probabilities and Potentiel, chapters V-VIII, North-Holland, (1982).

[39] D.M. Delong. Crossing probabilities for a square root boundary for a Bessel process. *Comm. Statist A-Theory Methods*, 10:2197–2213, 1981.

[40] G. Di Nunno, T. Meyer-Brandis, B. Øksendal, and F. Proske. Optimal portfolio for an insider in a market driven by lévy processes. *Quantitative Finance*, 6:83–94, 2006.

[41] P. Ehlers and Ph. Schönbucher. Background filtrations and canonical loss processes for top-down models of portfolio credit risk. *Finance and Stochastics*, 13:79–103, 2009.

[42] N. El Karoui, M. Jeanblanc, and Y. Jiao. What happens after a default: the conditional density approach. *Stochastic Processes and their Applications*, 120(7):1011–1032, 2010.

[43] R.J. Elliott. *Stochastic Calculus and Applications.* Springer, Berlin, 1982.

[44] R.J. Elliott and M. Jeanblanc. Incomplete markets and informed agents. *Mathematical Method of Operations Research*, 50:475–492, 1998.

[45] R.J. Elliott, M. Jeanblanc, and M. Yor. On models of default risk. *Math. Finance*, 10:179–196, 2000.

[46] M. Emery. Espaces probabilisés filtrés : de la théorie de Vershik au mouvement Brownien, via des idées de Tsirelson. In *Séminaire Bourbaki, 53ième année*, volume 282, pages 63–83. Astérisque, 2002.

[47] A. Eyraud-Loisel. BSDE with enlarged filtration. option hedging of an insider trader in a financial market with jumps. *Stochastic Processes and their Appl.*, 115:1745–1763, 2005.

[48] J.D. Fermanian and O. Vigneron. On break-even correlation: the way to price structured credit derivatives by replication. *Preprint*, 2010.

[49] D. Filipovi´ c, L. Overbeck, and T. Schmidt. Dynamic CDO term structure. *Mathematical Finance*, 2010.

[50] D. Filipovic, L. Hughston, and Macrina A. Conditional density models for asset pricing. *Preprint*, 2010.

[51] J-P. Florens and D. Fougere. Noncausality in continuous time. *Econometrica*, 64:1195–1212, 1996.

[52] H. Föllmer and P. Imkeller. Anticipation cancelled by a Girsanov transformation: a paradox on Wiener space. *Ann. Inst. H. Poincaré, Prob. Stat.*, 26:569–586, 1993.

[53] H. Föllmer and Ph. Protter. Local martingales and filtration shrinkage. *ESAIM: Probability and Statistics*, 2011.

[54] H. Föllmer, C-T Wu, and M. Yor. Canonical decomposition of linear transformations of two independent Brownian motions motivated by models of insider trading. *Stochastic Processes and their Appl.*, 84:137–164, 1999.

[55] C. Fontana, M. Jeanblanc, and S. Song. On arbitrages arising with honest times. *Finance and Stochastics*, 2014.

[56] D. Gasbarra, E. Valkeika, and L. Vostrikova. Enlargement of filtration and additional information in pricing models: a Bayesian approach. In Yu. Kabanov, R. Lipster, and J. Stoyanov, editors, *From Stochastic Calculus to Mathematical Finance: The Shiryaev Festschrift*, pages 257–286, 2006.

[57] A. Grorud and M. Pontier. Insider trading in a continuous time market model. *International Journal of Theoretical and Applied Finance*, 1:331–347, 1998.

[58] A. Grorud and M. Pontier. Asymmetrical information and incomplete markets. *International Journal of Theoretical and Applied Finance*, 4:285–302, 2001.

[59] S-W. He, J-G. Wang, and J-A. Yan. *Semimartingale Theory and Stochastic Calculus.* CRC Press Inc, 1992.

[60] C. Hillairet. Comparison of insiders' optimal strategies depending on the type of side-information. *Stochastic Processes and their Appl.*, 115:1603–1627, 2005.

[61] P. Imkeller. Random times at which insiders can have free lunches. *Stochastics and Stochastics Reports*, 74:465–487, 2002.

[62] P. Imkeller. Malliavin's calculus in insider models: additional utility and free lunches. *Mathematical Finance*, 13:153Ũ169, 2003.

[63] P. Imkeller, M. Pontier, and F. Weisz. Free lunch and arbitrage possibilities in a financial market model with an insider. *Stochastic Processes and their Appl.*, 92:103–130, 2001.

[64] J. Jacod. *Calcul stochastique et Problèmes de martingales*, volume 714 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1979.

[65] J. Jacod. Grossissement initial, hypothèse H' et théorème de Girsanov. In *Séminaire de Calcul Stochastique 1982-83*, volume 1118 of *Lecture Notes in Mathematics*. Springer-Verlag, 1987.

[66] F. Jamshidian. Valuation of credit default swap and swaptions. *Finance and Stochastics*, 8:343–371, 2004.

[67] M. Jeanblanc and Y. Le Cam. Progressive enlargement of filtration with initial times. *Stochastic Processes and their Applic.*, 119:2523–2543, 2009.

[68] M. Jeanblanc and M. Rutkowski. Modeling default risk: an overview. In *Mathematical Finance: Theory and Practice, Fudan University*, pages 171–269. Modern Mathematics Series, High Education press. Beijing, 2000.

[69] M. Jeanblanc and M. Rutkowski. Modeling default risk: Mathematical tools. *Fixed Income and Credit risk modeling and Management, New York University, Stern School of Business, Statistics and Operations Research Department, Workshop*, 2000.

[70] M. Jeanblanc and S. Song. Default times with given survival probability and their f-martingale decomposition formula. *Stochastic Processes and their Applic.*, 121:1389–1410, 2011.

[71] M. Jeanblanc and S. Song. Explicit model of default time with given survival probability. *Stochastic Processes and their Applic.*, 121:1678–1704, 2011.

[72] M. Jeanblanc, M. Yor, and M. Chesney. *Martingale Methods for financial Markets*. Springer Verlag, Berlin, 2007.

[73] Th. Jeulin. *Semi-martingales et grossissement de filtration*, volume 833 of *Lecture Notes in Mathematics*. Springer-Verlag, 1980.

[74] Th. Jeulin and M. Yor. Grossissement d'une filtration et semi-martingales : formules explicites. In C. Dellacherie, P-A. Meyer, and M. Weil, editors, *Séminaire de Probabilités XII*, volume 649 of *Lecture Notes in Mathematics*, pages 78–97. Springer-Verlag, 1978.

[75] Th. Jeulin and M. Yor. Nouveaux résultats sur le grossissement des tribus. *Ann. Scient. Ec. Norm. Sup.*, 11:429–443, 1978.

[76] Th. Jeulin and M. Yor. Inégalité de Hardy, semimartingales et faux-amis. In P-A. Meyer, editor, *Séminaire de Probabilités XIII*, volume 721 of *Lecture Notes in Mathematics*, pages 332–359. Springer-Verlag, 1979.

[77] Th. Jeulin and M Yor, editors. *Grossissements de filtrations: exemples et applications*, volume 1118 of *Lecture Notes in Mathematics*. Springer-Verlag, 1985.

[78] G. Kallianpur and C. Striebel. Estimation of stochastic systems: Arbitrary system process with additive white noise observation errors. *Ann. Math. Statist.*, 39(3):785–801, 1968.

[79] I. Karatzas and I. Pikovsky. Anticipative portfolio optimization. *Adv. Appl. Prob.*, 28:1095–1122, 1996.

[80] Y. Kchia, M. Larson, and Ph. Protter. Linking progressive and initial filtration expansions. *arkiv*, 2014.

[81] Y. P Kchia. *Semimartingales et Problématiques Récentes en Finance Quantitative*. Thèse de doctorat, Ecole Polytechnique, 2012.

[82] M. Keller-Ressel, A. Papapantoleon, and J. Teichman. The affine libor models. *Preprint*, 2010.

[83] A. Kohatsu-Higa. Enlargement of filtrations and models for insider trading. In J. Akahori, S. Ogawa, and S. Watanabe, editors, *Stochastic Processes and Applications to Mathematical Finance*, pages 151–166. World Scientific, 2004.

[84] A. Kohatsu-Higa. Enlargement of filtration. In *Paris-Princeton Lecture on Mathematical Finance, 2004*, volume 1919 of *Lecture Notes in Mathematics*, pages 103–172. Springer-Verlag, 2007.

[85] A. Kohatsu-Higa and B. Øksendal. Enlargement of filtration and insider trading. *Preprint*, 2004.

[86] D. Kreher. Thèse de doctorat, Univ. Zurich, 2014.

[87] H. Kunita. Some extensions of Itô's formula. In J. Azéma and M. Yor, editors, *Séminaire de Probabilités XV*, volume 850 of *Lecture Notes in Mathematics*, pages 118–141. Springer-Verlag, 1981.

[88] S. Kusuoka. A remark on default risk models. *Adv. Math. Econ.*, 1:69–82, 1999.

[89] D. Lando. On Cox processes and credit risky securities. *Review of Derivatives Research*, 2:99–120, 1998.

[90] L. Li. *Random Times and Enlargements of Filtrations*. Thèse de doctorat, Sydney University, 2012.

[91] R.S.R. Liptser and A.N. Shiryaev. *Statistics of Random Processes*. Springer, 2nd printing 2001, 1977.

[92] D. Madan, B. Roynette, and M. Yor. An alternative expression for the Black-Scholes formula in terms of Brownian first and last passage times. *Preprint, Université Paris 6*, 2008.

[93] R. Mansuy and M. Yor. *Random Times and (Enlargement of) Filtrations in a Brownian Setting*, volume 1873 of *Lectures Notes in Mathematics*. Springer, 2006.

[94] H. Matsumoto and M. Yor. A relationship between brownian motions with opposite drifts via certain enlargements of the brownian filtration. *Osaka J. Math.*, 38:383–398, 2001.

[95] G. Mazziotto and J. Szpirglas. Modèle général de filtrage non linéaire et équations différentielles stochastiques associées. *Ann. Inst. H. Poincaré*, 15:147–173, 1979.

[96] P-A. Meyer. Sur un problème de filtration. In *Séminaire de Probabilités VII*, volume 321 of *Lecture Notes in Mathematics*, pages 223–247. Springer-Verlag, 1973.

[97] A. Nikeghbali. An essay on the general theory of stochastic processes. *Probability Surveys*, 3:345–412, 2006.

[98] A. Nikeghbali and M. Yor. A definition and some properties of pseudo-stopping times. *The Annals of Probability*, 33:1804–1824, 2005.

[99] A. Nikeghbali and M. Yor. Doob's maximal identity, multiplicative decompositions and enlargements of filtrations. In D. Burkholder, editor, *Joseph Doob: A Collection of Mathematical Articles in his Memory*, volume 50, pages 791–814. Illinois Journal of Mathematics, 2007.

[100] J.W. Pitman and M. Yor. Bessel processes and infinitely divisible laws. In D. Williams, editor, *Stochastic integrals, LMS Durham Symposium, Lect. Notes 851*, pages 285–370. Springer, 1980.

[101] Ph. Protter. *Stochastic Integration and Differential Equations*. Springer, Berlin, Second edition, 2005.

[102] L.C.G. Rogers and D. Williams. *Diffusions, Markov processes and Martingales, Vol 1. Foundations*. Cambridge University Press, Cambridge, second edition, 2000.

[103] P. Salminen. On last exit decomposition of linear diffusions. *Studia Sci. Math. Hungar.*, 33:251–262, 1997.

[104] M.J. Sharpe. Some transformations of diffusion by time reversal. *The Annals of Probability*, 8:1157–1162, 1980.

[105] S. Song. *Grossissement de filtration et problèmes connexes*. Thèsis, Paris IV University, 1997.

[106] S. Song. Optional splitting formula in a progressively enlarged filtration. *ESAIM Probability and Statistics*, 2013.

[107] C. Stricker and M. Yor. Calcul stochastique dépendant d'un paramètre. *Z. Wahrsch. Verw. Gebiete*, 45(2):109–133, 1978.

[108] Ch. Stricker. Quasi-martingales, martingales locales, semimartingales et filtration naturelle. *Z. Wahr. Verw. Gebiete*, 39:55–63, 1977.

[109] B. Tsirel'son. Within and beyond the reach of Brownian innovations. In *Proceedings of the International Congress of Mathematicians*, pages 311–320. Documenta Mathematica, 1998.

[110] D. Williams. A non-stopping time with the optional-stopping property. *Bull. London Math. Soc.*, 34:610–612, 2002.

[111] C-T Wu. *Construction of Brownian Motions in Enlarged Filtrations and Their Role in Mathematical Models of Insider Trading*. Thèse de doctorat, Humbolt University, 1999.

[112] Ch. Yoeurp. Grossissement de filtration et théorème de Girsanov généralisé. In Th. Jeulin and M. Yor, editors, *Grossissements de filtrations: exemples et applications*, volume 1118. Springer, 1985.

[113] M. Yor. Grossissement d'une filtration et semi-martingales : théorèmes généraux. In C. Dellacherie, P-A. Meyer, and M. Weil, editors, *Séminaire de Probabilités XII*, volume 649 of *Lecture Notes in Mathematics*, pages 61–69. Springer-Verlag, 1978.

[114] M. Yor. Grossissement de filtrations et absolue continuité de noyaux. In Th. Jeulin and M. Yor, editors, *Grossissements de filtrations: exemples et applications*, volume 1118. Springer, 1985.

[115] M. Yor. *Some Aspects of Brownian Motion, Part II: Some Recent Martingale Problems.* Lectures in Mathematics. ETH Zürich. Birkhäuser, Basel, 1997.

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